Development of the Boundary Element Method Utilizing the Discrete Integral Method

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Graduate School of Marine Science and Engineering
Nagasaki University

Huang Chao

CONTENTS

CHAPTER 1

GENERAL INTRODUCTION

1.1 INTRODUCTION

Once a mathematical model of any kind of engineering problem has been constructed, efforts are directed towards obtaining a solution of equations. The region is often a very complicated shape and composed of zones of different materials with complex properties. Since the governing equation has different form and various conditions are specified on the boundaries, numerical methods become the effective means of obtaining adequately precise and detailed results.

The boundary element method (BEM) [1-1]~[1-2] is a well established technique for the analysis of engineering problems, particularly those involving linear analysis. The main advantage of BEM over other technique is the possibility of discretizing only the boundary of the problem instead of the whole domain, as required for instance by Finite Elements. Another attractive feature of BEM techniques is high accuracy. If domain integrals occur due to body forces, heat source, nonlinearity or time-dependent effects, generally, these integrals are carried out using cell elements [1-3].

In classical boundary element method, although the volume integrals are performed by discretizing the domain into cells and no further unknowns are needed, it is usually to require a integration over the whole body. In this case, the classical boundary element method becomes rather cumbersome and not only loses the advantage of a boundary element scheme, but also needs more computer space and time. Several methods have been developed to take domain integrals to the boundary in order to eliminate the need for internal cells. These methods will also be described in the section 1.2.

The main objective of this thesis is to establish a new approach, developed by my advisor Professor Kisu, in which distribution of the delta function is used. In this approach the function is approximated utilizing distribution of the delta function and the discrete integral method [1-4]~[1-5] is developed using this functional approximation. The discrete integral method is employed in the domain integral, general integral, bending problem of beam and steady-state heat conduction problem.

1.2 BACKGROUND OF RESEARCHES ON TREATMENT OF DOMAIN INTEGRALS IN BOUNDARY ELEMENT METHOD

BEM expresses the solution of a boundary value problem in terms of an integral equation, which is then solved numerically. For many engineering problems this equation contains boundary integrals only. This means that only the boundary needs to be discretized. However, for some problems, e.g. described by the Poisson type differential equation, both domain and boundary integrals appear in the integral equation. Although the domain integral does not introduce any new unknowns it requires additional effort to generate internal cells needed for numerical integration over a whole domain. This approach becomes numerically inefficient.

Several methods have been proposed to convert domain integrals occurring in BEM into equivalent boundary form. The first publication on this subject was due to T. A. Cruse [1-6] in 1975. The article was concerned with the problem of elastic fracture mechanics which body force exists and developed a technique, called Galerkin Tensor Method. When body force is a constant and linear loads, a good numerical results were obtained. It was found that the variation of the constant in the fundamental solution affects the numerical values given by Galerkin Tensor Method. Later, different authors [1-3], [1-7]~[1-13] employed this method in elastic problems.

In 1981 Kirchhoff's transformation was presented by Yu. N. Akkuratov V. N. Mikhailov [1-14] and R. Bialecki, A. Nowak [1-15]. The steady state nonlinear heat conduction equation with temperature dependence of thermal conductivity was described. The use of Kirchhoff's transformation was shown to be convenient when the thermal conductivity dependences the temperature. This transformation transfers the nonlinearity from differential equation and boundary conditions only into boundary conditions of the third kind. When the prescribed boundary conditions are of the first and second kind only, the problem becomes linear and can be solved by using the classical BEM formulation. Generally, a system of nonlinear equations has to be solved. This approach presents several advantages over methods where effects of nonlinearity are treated as additional heat sources. Heat radiation and temperature dependent heat transfer coefficient were also discussed. It was found that direct iteration procedures diverge when heat radiation plays a considerable role. A variant of the incremental method to solve such problems was developed.

One year latter, C. A. Brebbia and D. Nardini [1-16] proposed a more general technique which takes domain integrals to the boundary and is now called dual reciprocity method (DRM). They first applied it to the study of dynamical problems. The nonlinear term of the equation can be expressed using a series of approximating functions. The approximating functions were chosen as the function of the distance between pre-specified fixed collocation points and a field point. To reduce the domain integral in the equation to equivalent boundary integrals, a new auxiliary non-homogeneous Laplacian field is defined. In 1986 L. C. Wrobel, D. Nardini and C. A. Brebbia [1-17] applied this approach to the problem of transient heat conduction. In 1993 B. A. Davis, and P. J. Gramman, et al [1-18] used this DRM for the problems of simulating flow and heat transfer. In this research, the solution of the equation of energy for flow problems was presented. To solve velocities and velocity gradients in convection and viscous dissipation problems, the boundary integral equation for creeping flow were presented as well. The nonlinear terms were treated with DRM with randomly distributed internal nodes. Algorithms for heat generation during exothermic cure reaction, viscous dissipation, convection, and viscous dissipation combined with convection were developed. Oscillations in the convection problems were avoided by increasing the number of elements on the boundary. Heat transfer problems involving more complicated fluid flow were computed using an existing boundary element code.

In 1988, an efficient technique for reducing domain integrals to the boundary was developed by J. P. S. Azevedo and C. A. Brebbia [1-19] . In this paper, the integral corresponding to any sources acting on an internal region can be replaced by equivalent sources distributed on the boundary of that region by using particular solutions. The effect of these equivalent sources is taken to the boundary by the usual type of influence coefficients without introducing new unknowns to the problems. The approach consists of two different steps. The first step is to express the source distribution in terms of a linear combination of functions for which particular solutions are known. In many cases this is straight forward as the expressions of the source terms are given exactly in terms of polynomials or Fourier series. This step amounts to calculating a set of equivalent potential and flux source distributions on the boundary of the region with source distribution. The second step is to take the influence of the new set of equivalent sources to the boundary.

In 1989, A. J. Nowak and C. A. Brebbia [1-20] developed a new technique, called the Multiple Reciprocity Method (MRM), and applied it to solve Poisson and Helmholtz equations. In this method, the behavior of a source function inside the domain is represented by a series of its derivatives, or more precisely, by the series of subsequent Laplacians of source function calculated on the boundary. The series of Laplacians is defined by the recurrence formula. MRM leads to an exact integral equation. The simplifications are introduced as the stage of discretization of the boundary into boundary elements and due to truncation of MRM series. When the series converges its value can be calculated with high accuracy and as a consequence, results of MRM are also very accurate. In 1991, A. C. Neves and C. A. Brebbia [1-21] employed in the problem of elasticity. In this research, it has been extended to solve a completely different problem, i.e. the Navier equations of elasticity. The governing equations are much more complex than for the case of the generalized Poisson's equation and it has been necessary to develop higher order fundamental solutions which are not available in the literature. In addition, the resulting integrals were computed numerically using a third order polynomial transformation and a selective numerical integration scheme.

In 1996, Kamiya, N. [1-23] developed a method by the use of the new variable and applied it to the steady state nonlinear heat conduction equation with temperature dependence of thermal conductivity. The linear governing equation of the new variable was obtained when thermal conductivity was described by linear, exponential and power functions in terms of temperature.

1.3 THE PRESENT WORK

In order to avoid using the higher order fundamental solutions and the derivatives of the heat source in MRM, the discrete integral method is developed for the domain integral. At first the function is approximated utilizing distribution of the delta function. Then using this functional approximation the discrete integral method is established. This discrete integral method is employed in bending problem of beam and steady-state heat conduction problem. In the nonlinear heat conduction problem the discrete integral method avoids the complex inverse transform using the Kirchhoff's transformation, and the limitation of three cases of the thermal conductivity using a method by the use of the new variable.

1.3.1 Approximation of function and the discrete integral method utilizing distribution of the delta function

In this thesis, approximation of function utilizing distribution of the delta function in BEM is proposed. The function is approximated by distribution of the delta function. Using Green's identity, the function is carried out by the boundary condition and strength of distribution of the delta function. Accuracy of this theory is examined by the known function. Using this functional approximation the discrete integral method utilizing distribution of the delta function is obtained. The discrete integral method replaces domain integrals by equivalent boundary equation and sum of some values at some points within the domain.

The discrete integral method is used for the two-dimensional heat conduction problem with a heat source. The domain integral is performed not using the internal cells.

When an external force exists, the domain integral is treated by the discrete integral method utilizing the delta function for bending problems of continuous and inhomogeneous beam. This approach brings about high efficiency on the calculations.

The steady-state nonlinear heat conduction equation with temperature dependence of thermal conductivity is discussed by the discrete integral method utilizing delta function. This approach replaces the governing equation by a new form. This form is suitable for any kind of temperature dependence of heat conductivity. The theoretical view of the discrete integral approach and its validity through one-dimensional example is investigated.

1.3.2 The outline of this thesis

The thesis is composed of six chapters which are as follows.

In chapter 1, a general introduction of background of researches on treatment of domain integrals in BEM is presented.. At last, the present work is described.

In chapter 2, at first, the kind and characteristic of BEM are presented. Then one-dimensional and two dimensional Poisson equations are converted into integral equations using the direct method. By applying fundamental solution, the complete formulation are obtained. At last, the numerical implementation is discussed in detail.

In chapter 3, the basic theory of the functional approximation utilizing distribution of the delta function is proposed. From this approximation of function the discrete integral method is developed. Some numerical examples are submitted to verify the validity of this new method.

In chapter 4, a new analysis method for bending problem of beam is proposed. A scheme without any variables at intermediate points is established. A generalized solution scheme for an inhomogeneous beam is obtained. The domain integral is evaluated by the discrete integral method utilizing the delta function.

In chapter 5, the steady state nonlinear heat conduction equation with temperature dependence of thermal conductivity is solved by the discrete integral method. This approach is applicable to general temperature dependence of heat conductivity.

In chapter 6, the conclusions of this work are summarized.

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CHAPTER 2

BASIC THEORY OF THE BOUNDARY ELEMENT METHOD

2.1 INTRODUCTION

The chapter is primarily intended to show what is the characteristic of the boundary element method and how the governing equation with prescribed boundary conditions can be converted into a suitable integral equation using the direct boundary element method. Next, the complete boundary formulation using the fundamental solution is obtained. The numerical implementation is discussed in detail.

Numerical methods for engineering have been investigated by many researchers for many years. These methods can be classified in three main categories, finite differences, finite elements, and boundary element methods. The finite difference [2-1]~[2-2] is the first successfully applied to numerical method and is usually derived by direct application of a difference operator corresponding to the governing differential equation. This operation is carried out at series of nodes within the domain of the body. This approach, however, possesses some drawbacks which are immediately apparent when complicated boundary geometries and relatively accurate solutions are attempted. The finite element method [2-3]~[2-4] is the most popular numerical method. The domain of the body is subdivided into a collection of connected subdomains, called finite elements. Polynomial functions are then chosen to locally approximate the actual behavior of the solution. A "best fit" for the approximation is then obtained through the variational principle. The method is more efficient than the finite difference approach and is applied to a very wide range of linear and nonlinear problems. There are, however, many classes of problems for which finite elements do not behave satisfactorily and this has led researchers to look for alternative techniques such as those based on integral equation.

In boundary element technique, the governing differential equation is transformed into boundary integral equation. If the boundary equation is integrated in closed form, this solution will be exact. But this is virtually impossible in practical problems and approximations have to be introduced. Therefore, the boundary may be discretized into a number of elements over which polynomial function are introduced to interpolate the values of the approximated solution between the nodal points. This allows for the evaluation of the relevant integrals, usually by some numerical process, resulting in a final system of equations.

In the followings the direct boundary element method is used for theoretical analysis. In this method , the unknown functions appearing in the integral equations are the actual physical variables of the problem. The internal potential and flux are directly computed afterwards using the boundary values obtained through the solution of the system of equations.

2.2 THE KINDS AND MAIN CHARACTERISTICS OF BEM [2-5]~[2-6]

Although all BEM have a common origin they are divided naturally into three different but closely categories:

1. The direct method: the unknown functions appearing in the integral equations are the actual physical variables of the problem such as potential and its flux. Thus, for example, in an potential problem such an integral equation solution would yield all the fluxes and potentials on the boundary directly and those within the domain can be derived from the boundary values by numerical integration.
2. Semi-direct method: the integral equations can be formulated in terms of unknown function analogous to stress functions in elasticity or stream functions in potential flow.

When the solution has been obtained in these terms, simple differentiation will yield.

3. The indirect method: the integral equations are expressed entirely in term of a unit singular solution of the original differential equations distributed at a specific density over the boundaries of the region of interest. The density functions themselves have no specific physical significance but once they have been obtained from a numerical solution of the integral equations, the values of the solution parameters anywhere within the body can be calculated from them by simple integration processes.

The advantage of the boundary element method is summarized as follows:

1. BEM treats problems by single boundary discretization. This leads to a very much smaller system of simultaneous equations than any scheme of whole-body discretion used in FEM and FDM. So BEM can reduce the dimensionality of practical problems. For example, for two-dimensional problems, the method can generate a one-dimensional boundary integral equation and for three-dimensional problems only two-dimensional surface integral equation will arise. Because of the reduction in dimension, BEM can save the computer-memory volume, the calculating time and the work of data preparation. Thus for the great majority of practical problems, BEM can offer very substantial advantages over the other numerical methods.

2. By using influence solutions, the solution of unbounded problems does not require any special treatment. This implies that a large number of cases where the domain under consideration extends to infinity can be solved without resource to large meshes and artificial body conditions.

3. In BEM, the numerical result has high degree of accuracy. This method is suitable for problems such as stress concentration and those with high temperature gradient region.

4. Contrary to domain techniques, boundary element analysis codes are easy to interface with standard CAE systems.

2.3 TWO-DIMENSIONAL POISSON EQUATION [2-7]~[2-12]

2.3.1 The integral equation

The governing differential equation in two dimensions is given as

$$\nabla^2 u+f=0 \qquad \text{in } \Omega \tag{2.1}$$

where f is now a given distribution of source strengths over the domain Ω. The flux at any point will be $q=\dfrac{\partial u}{\partial n}$.

The boundary conditions for Eq. (2.1) as shown in Fig. 2.1 are written as

$$u=\bar{u} \qquad \text{on } \Gamma_1 \tag{2.2}$$

$$q=\frac{\partial u}{\partial n}=\bar{q} \qquad \text{on } \Gamma_2 \tag{2.3}$$

The source point is all expressed by p (or P, if on the boundary) and the integral point (observation point) by Q.

In order to investigate the possibility of actually integrating Eq. (2.1), over the range Ω, a function $u^*(x,p)$ will be introduced, which is, as yet, undefined except that it is

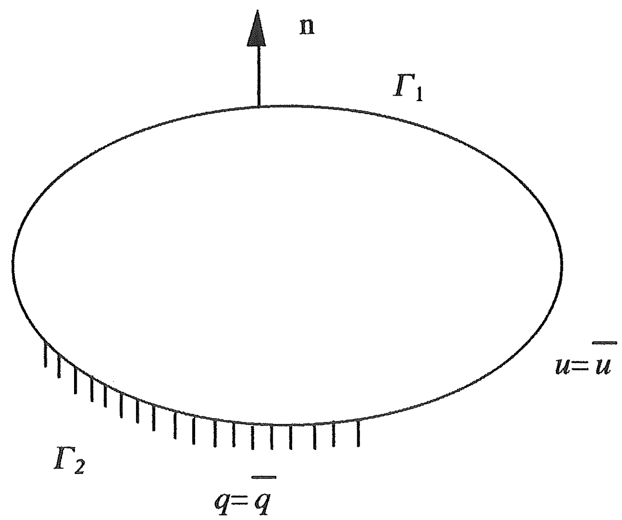

Figure 2.1: Potential problems in two dimentions

sufficiently continuous to be differentiable as often as required. If both sides of Eq. (2.1) are multiplied by $u^*(x,p)$ and integrated by parts, the following equation is obtained

$$\int_\Omega \left(\nabla^2 u(Q)+f(Q)\right)u^*(Q,p)d\Omega = 0 \qquad (2.4)$$

Therefore,

$$\int_\Omega \left(\nabla^2 u(Q)+f(Q)\right)u^*(Q,p)d\Omega = \int_\Gamma \frac{\partial u(Q)}{\partial n}u^*(Q,p)d\Gamma - \int_\Omega \left(\nabla u(Q)\right)\nabla u^* d\Omega$$

$$+\int_\Omega f(Q)u^*(Q,p)d\Omega\ (Q)$$

$$=\int_\Gamma \frac{\partial u(\mathrm{Q})}{\partial n}u^*(Q,p)d\Gamma - \int_\Gamma u(Q)\frac{\partial u^*(Q,p)}{\partial n}d\Gamma$$

$$+\int_\Gamma u(Q)\nabla^2 u^*(Q,p)d\Omega(Q) + \int_\Omega f(Q)u^*(Q,p)d\Omega\ (Q) = 0 \qquad (2.5)$$

Eq. (2.4) can be written as

$$\int_\Gamma \frac{\partial u(\mathrm{Q})}{\partial n}u^*(Q,p)d\Gamma - \int_\Gamma u(Q)\frac{\partial u^*(Q,p)}{\partial n}d\Gamma$$

$$+\int_\Gamma u(Q)\nabla^2 u^*(Q,p)d\Omega(Q) + \int_\Omega f(Q)u^*(Q,p)d\Omega\ (Q)=0 \qquad (2.6)$$

Now $u^*(x,p)$ is specified to be a solution of

$$\nabla^2 u^*(Q,p) = -\delta(Q-p) \tag{2.7}$$

Here, $\delta(Q-p)$ is the delta function. (To see appendix in detail). The fundamental solution which satisfies the Eq. (2.7) is obtained as

$$u^*(Q,p) = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right) \tag{2.8}$$

where r is the distance between Q and p. By Eqs (2.6) and (2.7) the following equation is obtained, when written out fully,

$$u(p) = \int_\Gamma \frac{\partial u(Q)}{\partial n} u^*(Q,p)d\Gamma - \int_\Gamma u(Q)\frac{\partial u^*(Q,p)}{\partial n}d\Gamma + \int_\Omega f(Q)u^*(Q,p)d\Omega\,(Q) \tag{2.9}$$

If the following formula is defined

$$q^*(Q,p) = \frac{\partial u^*(Q,p)}{\partial n} \tag{2.10}$$

then Eq. (2.9) can be written more concisely as

$$u(p) = \int_\Gamma u^*(Q,p)q(Q)d\Gamma(Q) - \int_\Gamma q^*(Q,p)u(Q)d\Gamma(Q) + \int_\Omega f(Q)u^*(Q,p)d\Omega\,(Q) \tag{2.11}$$

If the point p is imagined to approach the boundary Γ from inside Ω , Eq. (2.11) becomes

$$C(p)u(p) = \int_\Gamma u^*(Q,p)q(Q)d\Gamma(Q) - \int_\Gamma q^*(Q,p)u(Q)d\Gamma(Q)$$

$$+\int_\Omega f(Q)u^*(Q,p)d\Omega\,(Q) \tag{2.12}$$

where $C(p)$ is the position coefficient. If the point p is located on a boundary, $C(p)=\frac{1}{2}$.

If the point p is located inside domain, $C(p)=1$. Eq. (2.12) can calculate the potential at any point p from a knowledge of both the potential and the flux at all points around the boundary Γ and the specified internal source distribution.

2.3.2 Numerical implementation

In this section a general numerical procedure for the solution of boundary value problems will be described. Instead of attempting closed form solutions to Eq. (2.12), which is a difficult, a suitable numerical approach is here employed. The basic steps are summarized below:

(Here, N is the number of boundary nodes.)

1 . The boundary is discretized into a series of elements over which the potential and the flux are chosen to be piecewise interpolated between the element nodal points;

2 . Eq. (2.12) is applied in discretized form to each nodal point of the boundary and the integrals are computed over each boundary element. A system of N linear algebraic equation involving the set of N the nodal flux and N nodal potential is therefore obtained;

3 . Boundary conditions are imposed and consequently N nodal values are prescribed. The system of N equations can therefore be solved by standard methods to obtain the remaining boundary data.

If the boundary is divided into NE cells and the domain is divided into M cells, Eq. (2.12) can be written as

$$c(p)u(p)+\sum_{i=1}^{NE}\int_{\Gamma_i} q^*(Q,p)u(Q)d\Gamma(Q)=\sum_{i=1}^{NE}\int_{\Gamma_i} u^*(Q,p)q(Q)d\Gamma(Q)$$

$$+\sum_{i=1}^{M}\int_{\Omega_i} f(Q)u^*(Q,p)d\Omega(Q) \tag{2.13}$$

In this paper quadratic shape functions as shown in Fig. 2.2 are employed. The coordinates of points located within each element Γ_i are expressed in terms of interpolation functions and the nodal coordinates of the element by the following relation

$$x(\xi)=\Phi^1(\xi)x^1+\Phi^2(\xi)x^2+\Phi^3(\xi)x^3 \tag{2.14}$$

where ξ is an intrinsic coordinate defined on boundary element, which varies between -1 and 1. $\Phi^1(\xi)$, $\Phi^2(\xi)$ and $\Phi^3(\xi)$ are expressed as

$$\Phi^1(\xi)=\frac{1}{2}\xi(\xi-1),\quad \Phi^2(\xi)=(1-\xi)(1+\xi),\quad \Phi^3(\xi)=\frac{1}{2}\xi(\xi+1)$$

In similar way, boundary potential and flux are approximated over element through interpolation functions

$$u(\xi)=\Phi^1(\xi)u_j^1+\Phi^2(\xi)u_j^2+\Phi^3(\xi)u_j^3 \tag{2.15}$$

$$q(\xi)=\Phi^1(\xi)q_j^1+\Phi^2(\xi)q_j^2+\Phi^3(\xi)q_j^3 \tag{2.16}$$

where u_j^1, u_j^2, u_j^3, q_j^1, q_j^2 and q_j^3 contain the nodal potential and flux respectively. In order to calculate Eq. (2.13), it is necessary to transform the boundary element $d\Gamma$ from the global system to this intrinsic system of coordinates

$$d\Gamma=\sqrt{\left(\frac{dx_1}{d\xi}\right)^2+\left(\frac{dx_2}{d\xi}\right)^2}\,d\xi=G(\xi)d\xi \tag{2.17}$$

The boundary integral of Eq. (2.13) for i-th source point can be calculated by Eqs (2.15) ~

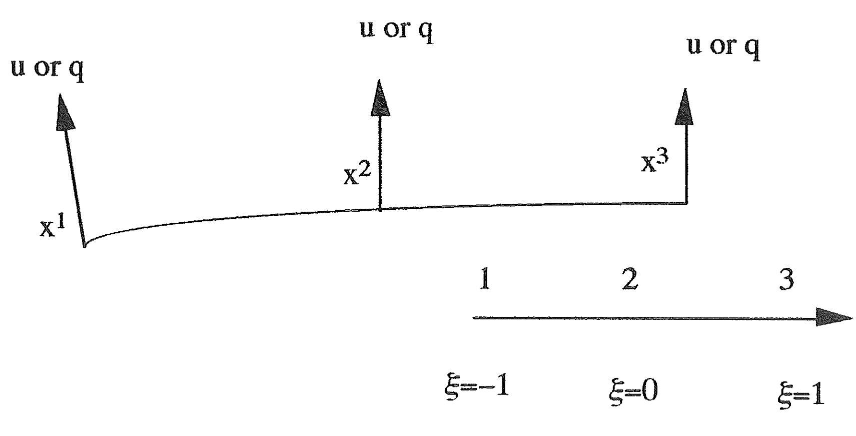

Figure 2.2: The quadratic shape functions and the homogenous coordinate

(2.17). The integrals in equation can be replaced by summations of the form.

$$\int_{\Gamma_i} q^*(Q,p)u(Q)d\Gamma(Q) = \sum_{k=1}^{3} \int_{-1}^{1} q^*(\xi)\Phi^k(\xi)u^k G(\xi)d\xi \tag{2.18}$$

$$\int_{\Gamma_i} u^*(Q,p)q(Q)d\Gamma(Q) = \sum_{k=1}^{3} \int_{-1}^{1} u^*(\xi)\Phi^k(\xi)q^k G(\xi)d\xi \tag{2.19}$$

The discretization of the boundary integrals has been discussed and next, emphasis will be given to the domain integrals of the source strengths by the same procedure. In this step the quadratic surface element as shown in Fig. 2.3 is used to model the geometry and the variations of the surface potential and flux respectively. For the domain discretizations of Eq. (2.13) the coordinates of points located within each cell Ω_i can be repressed by the following equation.

$$x(\xi) = \sum_{k=1}^{8} \Phi^k(\xi)x^k \tag{2.20}$$

where $\xi\left(\xi_{1,}\xi_2\right)$ is an intrinsic coordinate defined on domain. x^k (k=1,2 ···, 8) The coordinates of some special points which define the geometry of the cell. $\Phi^k(\xi)$ (k=1,2 ···, 8) represents the interpolation functions.

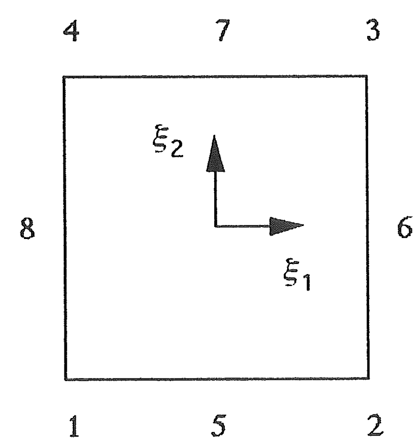

Figure 2.3: The quadratic surface element

$$\Phi^1=\frac{1}{4}(1-\xi_1)(1-\xi_2)(-\xi_1-\xi_2-1)$$

$$\Phi^2=\frac{1}{4}(1+\xi_1)(1-\xi_2)(\xi_1-\xi_2-1)$$

$$\Phi^3=\frac{1}{4}(1+\xi_1)(1+\xi_2)(\xi_1+\xi_2-1)$$

$$\Phi^4=\frac{1}{4}(1-\xi_1)(1+\xi_2)(-\xi_1+\xi_2-1)$$

$$\Phi^5=\frac{1}{2}(1-\xi_1)(1+\xi_1)(1-\xi_2)$$

$$\Phi^6=\frac{1}{2}(1+\xi_1)(1-\xi_2)(1+\xi_2)$$

$$\Phi^7=\frac{1}{2}(1-\xi_1)(1+\xi_1)(1+\xi_2)$$

$$\Phi^8=\frac{1}{2}(1-\xi_1)(1-\xi_2)(1+\xi_2)$$

In a similar way, the potential and flux are written as

$$u(\xi)=\sum_{k=1}^{8}\Phi^k(\xi)u^k \tag{2.21}$$

$$q(\xi)=\sum_{k=1}^{8}\Phi^k(\xi)q^k \tag{2.22}$$

It is convenient to compute the cell integrals by using suitable numerical quadrature scheme. The domain integral of Eq. (2.13) can be evaluated.

$$\int_{\Omega_i} f(Q)u^*(Q,p)d\Omega\,(Q)=\iint u^*(\xi)\left(\sum_{k=1}^{8}\Phi^k(\xi)f^k\right)\left|\widetilde{G(\xi)}\right|d\xi_1 d\xi_2 \tag{2.23}$$

where $\widetilde{G(\xi)}=\left|\frac{\partial x}{\partial \xi_1}\times\frac{\partial x}{\partial \xi_2}\right|$. In this method, the domain needs to be discretized to compute the volume integrals and the surface of the domain is divided into a series of elements. This BEM is called the classical or conventional boundary element method.

If Eqs (2.15) ~ (2.17) and Eq. (2.23) are substituted into Eq. (2.13), the following equation can be obtained as

$$c_i u_i + \sum_{j=1}^{NE}\sum_{k=1}^{3} h_{ij}^k u_j^k = \sum_{j=1}^{NE}\sum_{k=1}^{3} g_{ij}^k q_j^k + \sum_{j=1}^{M}\sum_{k=1}^{8} d_{ij}^k f_j^k \tag{2.24}$$

where $h_{ij}^k = \int_{-1}^{1} q^*(\xi)\Phi^k(\xi)G(\xi)d\xi$, $g_{ij}^k = \int_{-1}^{1} u^*(\xi)\Phi^k(\xi)G(\xi)d\xi$ and

$d_{ij}^k = \iint u^*(\xi)\Phi^k(\xi)\left|\widetilde{G(\xi)}\right| d\xi_1 d\xi_2$.

Eq. (2.24) for i-th source point can also be expressed as.

$$c_i u_i + \begin{bmatrix} h_{i1} & h_{i2} & \cdot & \cdot & \cdot & h_{iN} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \cdot \\ \cdot \\ \cdot \\ u_N \end{Bmatrix} = \begin{bmatrix} g_{i1} & g_{i2} & \cdot & \cdot & \cdot & g_{iN} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \cdot \\ \cdot \\ \cdot \\ q_N \end{Bmatrix} + \begin{bmatrix} \mathrm{b}_i \end{bmatrix} \tag{2.25}$$

From the application of Eq. (2.24) to all boundary nodes, a final system of equations for i-th source point arises

$$\begin{bmatrix} c_1+h_{11} & h_{12} & \cdots & h_{1N} \\ h_{21} & c_2+h_{22} & \cdots & h_{2N} \\ & & & \\ h_{N1} & h_{N2} & \cdots & c_N+h_{NN} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ \\ u_N \end{Bmatrix} =$$

$$\begin{bmatrix} g_{11} & g_{12} & \cdots & g_{1N} \\ g_{21} & g_{22} & \cdots & g_{2N} \\ & & & \\ g_{N1} & g_{N2} & \cdots & g_{NN} \end{bmatrix} \begin{Bmatrix} q_1 \\ q_2 \\ \\ q_N \end{Bmatrix} + \begin{Bmatrix} \mathrm{b}_1 \\ \mathrm{b}_2 \\ \\ \mathrm{b}_N \end{Bmatrix} \tag{2.26}$$

By applying the specified boundary conditions, Eq. (2.26) can be reordered and a set of simultaneous linear equations is obtained as

$$[A]\{X\}=\{f\} \tag{2.27}$$

where $[A]$ is a fully populated matrix. Vector $\{X\}$ is formed by the unknown potential and flux and the contribution of the prescribed values is included into vector f.

2.3.3 Calculation of the diagonal term using the rigid body mode.

The diagonal terms h_{ii} of the coefficient matrix on the left hand of the Eq. (2.26) can be evaluated indirectly using the rigid body mode. In the rigid body mode, the stress and the body force in the domain are always zero if only the rigid displacement occurs. We suppose a rigid displacement is u_c, Eq. (2.12) becomes

$$C(P)u_c=-\int_{\Gamma} u_c q^*(Q,P)d\Gamma(Q) \tag{2.28}$$

or $$\left\{C(P)+\int_{\Gamma} q^*(Q,P)d\Gamma(Q)\right\}u_c=0$$

Because Eq. (2.28) is correct for any u_c, this equation has no relation to boundary condition. The following equation can be obtained as

$$C(P)=-\int_{\Gamma} q^*(Q,P)d\Gamma(Q) \tag{2.29}$$

Discretizing the Eq. (2.29) yields

$$c_i+\sum_{j=1}^{NE}\int_{\Gamma_j} Q^*(Qp^i)d\Gamma\ (Q) = c_i+\sum_{j=1}^{N}h_{ij} \tag{2.30}$$

The diagonal terms in the matrix on the left hand of Eq. (2.26) can be calculated using the following equation.

$$c_i+h_{ii}=-\sum_{j=1(i\neq j)}^{N} h_{ij} \tag{2.31}$$

Eq. (2.31) provide useful means of computing the leading diagonal submatrices, avoiding analytical evaluation of the coefficients and the principal value integrals.

2.4 CONCLUSIONS

In this chapter, the basic theory and main characteristic of the direct BEM are as follows:

1. Since BEM treats problems by the discretization of the surfaces of the body, BEM has advantages over FDM and FEM. BEM reduce in dimensions so that BEM can save the computer-memory volume and the work of data preparation. On the other hand, the numerical result has high degree of accuracy.
2. In the two-dimensional Poisson problem the governing equation is converted into boundary integral equation. The complete integral formulation is presented using the fundamental solution. Using the quadratic boundary elements and quadratic surface element that the formulation of the system matrices is obtained. The domain is divided into a series of surface cells and on each of these cells integrals are carried on.
3. Calculation of the diagonal term is performed using the rigid body mode.

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CHAPTER 3

APPROXIMATION OF FUNCTION AND DISCRETE INTEGRAL METHOD UTILIZING DISTRIBUTION OF THE DELTA FUNCTION

3.1 INTRODUCTION

A large range of problems can be solved using the boundary element method. BEM requires only the discretization of the surface of the body into a series of elements. This characteristic not only reduces the number of unknowns but also considerably simplifies the amount of data required to run a problem. However, some problems such as the bending problem of the beam, the heat conduction problem and others, usually require the domain to be divided into internal cells. (This BEM is usually called the classical or conventional boundary element method.) This greatly increases the amount of data preparation needed and causes BEM to lose its main advantage over domain type methods.

In an effort to avoid the internal discretization, a great amount of research has been carried out to find a general and efficient method to transform domain integrals into equivalent boundary integrals. A. Nowak and C. A. Brebbia [3-1] developed a new technique, called the multiple reciprocity method (MRM), and applied it to solve Poisson and Helmholtz equations. MRM consists of proposing a sequence of the higher order fundamental solutions which permit the application of Green's second identity to successive domain integral terms in an effort to transform them into the boundary. Although increasing the order of the fundamental solution, the resulting recurrence formula reduces the order of the heat source. On the other hand, Y. Ochiai [3-2] has also used the similar way for a steady thermal stress problem.

However, MRM needs higher order fundamental solutions and derivatives of the heat source. Since it is sometime difficult to evaluate these values, the application of this method is limited.

In the section the new functional approximation is developed utilizing distribution of the delta function. Using this approximation of function the method, called the discrete integral method, has been proposed to solve general integral and domain integral. Since the present method uses the lower order fundamental solution, it is simpler than MRM.

3.2 THE MULTIPLE RECIPROCITY BOUNDARY ELEMENT METHOD [3-1]

In this section MRM for treating domain integrals will be described. The problem under consideration is a steady-state potential field governed by Poisson equation

$$\nabla^2 u + \frac{1}{k} f = 0 \tag{3.1}$$

where u stands for the potential, k is the conductivity coefficient and the function f represents the heat source.

The region is considered Ω with its boundary Γ . The application of the reciprocity theorem allows to transform the problem (3.1) into the following integral equation [1], [2].

$$kC(p)u(p) + \int_\Gamma q^{*(0)}(Q,p)u(Q)d\Gamma(Q)$$

$$= \int_\Gamma u^{*(0)}(Q,p)q(Q)d\Gamma(Q) + \int_\Omega f^{(0)}(Q)u^{*(0)}(Q,p)d\Omega\,(Q) \tag{3.2}$$

where the fundamental solution $u^{*(0)}$ and $f^{(0)}$ satisfy the following differential equation

$$\nabla^2 u^{*(0)}(Q,p) = -\delta(Q-p) \tag{3.3}$$

$$f^{(0)} = f \tag{3.4}$$

and the $u^{*(0)}$ has the form:

$$u^{*(0)}(Q,p)=\frac{1}{2\pi}\ln\left(\frac{1}{r}\right) \tag{3.5}$$

The delta function acts at point p and r is a geometrical distance measured from that point. The superscript (0) in the fundamental solution and the heat source has been included for the notation purpose.

The flux q and $q^{*(0)}$ are defined as the following formula

$$q=-k\frac{\partial u}{\partial n} \tag{3.6}$$

$$q^{*(0)}=-k\frac{\partial u^{*(0)}}{\partial n} \tag{3.7}$$

where n stands for the outward normal derivative.

In addition to boundary integrals, Eq. (3.2) contains a domain integral, which has a general form as follows

$$D^{(0)}=\int_{\Omega} u^{*(0)} f^{(0)} d\Omega \tag{3.8}$$

The multiple reciprocity method is a method that transforms the domain integral (3.8) into the boundary. The procedure introduces a sequence of the higher order fundamental solutions defined by the recurrence formulae

$$\nabla^2 u^{*(j+1)}=u^{*(j)} \qquad j=0,1,2\ \cdots \tag{3.9}$$

$$q^{*(j+1)}=-k\frac{\partial u^{*}(j+1)}{\partial n} \tag{3.10}$$

as well as sequence of the heat source Laplacians

$$f^{(j)}=\nabla^2 f^{(j-1)} \qquad j=1,2,\ \cdots \tag{3.11}$$

$$w^{(j)}=-k\frac{\partial f^{(j)}}{\partial n} \tag{3.12}$$

In the term of Eqs (3.9)~ (3.12), a series of boundary integrals are obtained as

$$D^{(0)}=\frac{1}{k}\sum_{j=0}^{\infty}\int_{\Gamma}\left(u^{*(j+1)}w^{(j)}-q^{*(j+1)}f^{(j)}\right)d\Gamma \tag{3.13}$$

It is worth pointing out that Eq. (3.13) is the exact form of the primary domain integral (3.8) as no simplifications have been made so far.

Finally introducing Eq. (3.13) into (3.2), the boundary formulation of the problem (3.1) is obtained as follows:

$$kC(p)u(p)+\int_{\Gamma}q^{*(0)}(Q,p)u(Q)d\Gamma(Q)$$

$$=\int_{\Gamma}u^{*(0)}(Q,p)q(Q)d\Gamma(Q)-\frac{1}{k}\sum_{j=0}^{\infty}\int_{\Gamma}\left(u^{*(j+1)}w^{(j)}-q^{*(j+1)}f^{(j)}\right)d\Gamma \tag{3.14}$$

The obtained formulation does not contain any domain integral, so its discretization does not involve any internal cells. The main advantage of the BEM is fully preserved. However, in the multiple-reciprocity method, since the higher order fundamental solutions and the derivatives of the heat source are applied to convergence of solution, this approach is not generally used.

3.3 APPROXIMATION OF FUNCTION [3-3]~[3-5]

In order to avoid using the higher order fundamental solutions and the derivatives of the heat source, the new functional approximation is proposed utilizing distribution of the delta function. This approach will be described as follows.

3.3.1 Approximation of function utilizing the delta function

The function f can be approximated by the delta function (the point source) as follows:

$$\nabla^{n}f=\sum_{i=1}^{m}E_{i}\delta(x-x_{i}) \tag{3.15}$$

where E_i is the strength of the point source, x_i is the position of the point source and m is the number of the point source and its strength. If using the point source $(n=2)$, the approximation of the function f can be obtained as [1,2,3]

$$\nabla^2 f=\sum_{i=1}^{m} D_i \delta(x-x_i) \tag{3.16}$$

Multiplying the both sides of Eq. (3.16) by a function which has a property of $\nabla^2 Y^* = -\delta(x-x_i)$, the following equation is obtained:

$$c(s)f(s)=\int_{\Gamma} (\nabla f Y^* - f \nabla Y^*) \boldsymbol{n} d\Gamma - \sum_{i=1}^{m} D_i Y^*(x_i,s) \tag{3.17}$$

$c(s)$ is a constant. The function f at any point can be expressed by the quantities of boundary condition and m unknown point sources with the strength of D_i. If the function f is a known function, the accuracy of this approximation can be examined. From Eq. (3.17) the following equation is obtained

$$\sum_{i=1}^{m} D_i Y^*(x_i,s) = \int_{\Gamma} (\nabla f Y^* - f \nabla Y^*) \boldsymbol{n} d\Gamma - c(s)f(s) \tag{3.18}$$

In this case, a set of simultaneous equations can be constructed with respect to D_i by monitoring the known function f at m points. Using D_i which is obtained by this method, the function f can be calculated at any point as follows

$$c(p)f(p)=\int_{\Gamma} (\nabla f Y^* - f \nabla Y^*) \boldsymbol{n} d\Gamma - \sum_{i=1}^{m} D_i Y^*(x_i,p) \tag{3.19}$$

It is obvious that by this scheme, the function f can be identified.

3.3.2 Approximation of function utilizing the continuous distribution of the delta function

In Eq. (3.15) the function f is approximated by the point source and its strength. In the following the function f is approximated by the utilizing the continuous distribution of the delta function (the line source). On the line source, the strength of the line source is a continuous function. This approximation is expressed by the following equation and the line source is shown in Fig. 3.1.

$$\nabla^2 f=\sum_{i=1}^{m} L(S_i)\delta^L(S_i) \tag{3.20}$$

where S_i is the line source. $L(S_i)$ is the strength of the line source. $\delta^L(S_i)$ denotes the continuous distribution of the delta function on the line source. If using the line source as shown in Fig. 3.2, from Eq. (3.20) the approximation of the function f can be rewritten as

$$\nabla^2 f=\sum_{i=1}^{m} L(y_i)\delta^L(y_i) \tag{3.21}$$

Using the same method as utilizing the point source, $L(y_i)$ can be evaluated as the following equation

$$\sum_{i=1}^{m} \int_{-1}^{1} L(y_i)Y^* dx = \int_{\Gamma} (\nabla f Y^* - f \nabla Y^*) \boldsymbol{n} d\Gamma - c(s)f(s) \tag{3.22}$$

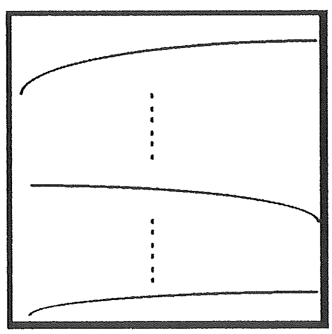


Figure 3.1: The general line source.

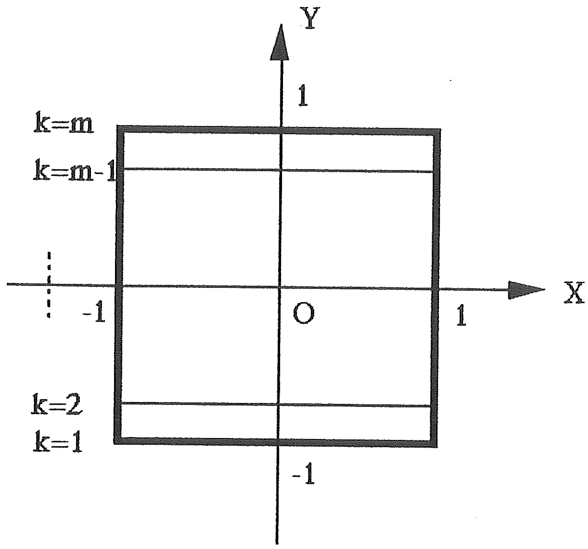


Figure 3.2: A model for line soures

Therefore, the function f at any point can be obtained by following equation.

$$c(p)f(p)=\int_{\Gamma}(\nabla f Y^{*}-f\nabla Y^{*})n d\Gamma-\sum_{i=1}^{m}\int_{-1}^{1}L(y_i)Y^{*}dx \qquad (3.23)$$

where p is any points.

3.3.3 Numerical examples

In this section, two examples are presented indicating the effectiveness of the functional approximation and the discrete integral method. These examples are:

1. Approximation of function utilizing the point source .
2. Approximation of function utilizing the line source.

Example 1

In this example, the functional approximation utilizing the point source described in the previous sections is employed in the two-dimensional problem as shown Fig. 3.3. The length of the square region is 2. The function $f(x, y)$ is chosen as $\sin(\frac{\pi y}{2})$. That is $f(x, y) = u(x, y) = \sin(\frac{\pi y}{2})$. The number of monitor point m $(m = 289)$ is distributed evenly on the square region.

From the table 3.1 (a), the results are compared with exact solution and they agree with each other 4 figures. But it is worth noting that when $y_{\to 0}$, the numerical results are not good and the maximum error is 24.29%. From table 3.1 (b) $y =$ 0.0625 (), the discrete integral method utilizing the point source does not suit for this problems. The numerical results are compared with the exact solution as shown in Figs 3.4 (a)~3.4 (h). It is evident that the discrete integral method utilizing the point source is not applicable to the general problems in two dimensions, Although this approach is employed in one-dimensional problem successfully. [3-3]~[3-9]

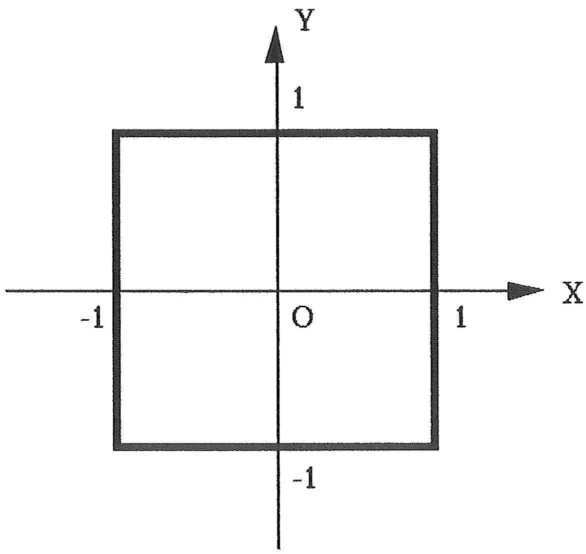


Figure 3.3: Square region of the functional approximation

Table 3.1 (a): The functional approximation

when $f(x, y) = \sin(\frac{\pi y}{2})$ and utilizing the point source.

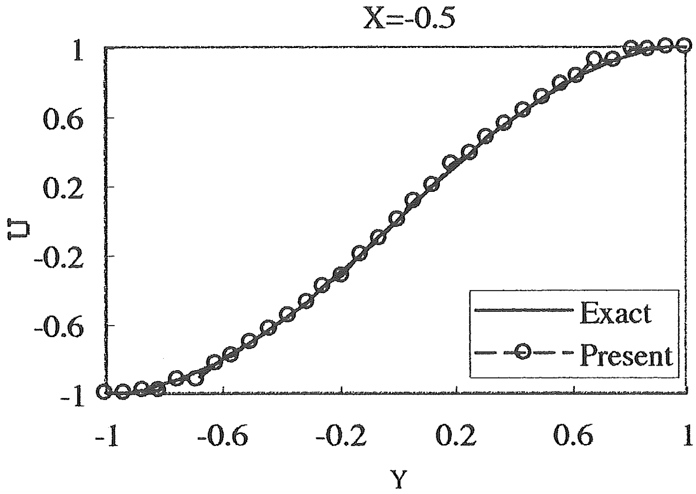
X	Y	EXACT	RESULTS
0.5	-0.5	-0.7071	-0.7071
0.5	0.5	0.7071	0.7071
0	0	0.0000	-2.77E-16
-0.5	-0.5	-0.7071	-0.7071
-0.5	0.5	0.7071	0.7071

Table 3.1 (b): The functional approximation

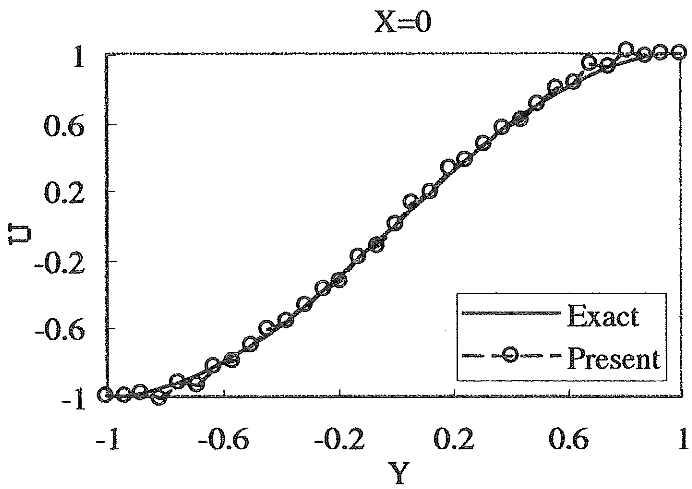
when $f(x, y) = \sin(\frac{\pi y}{2})$ and utilizing the point source.

Y=0.0625

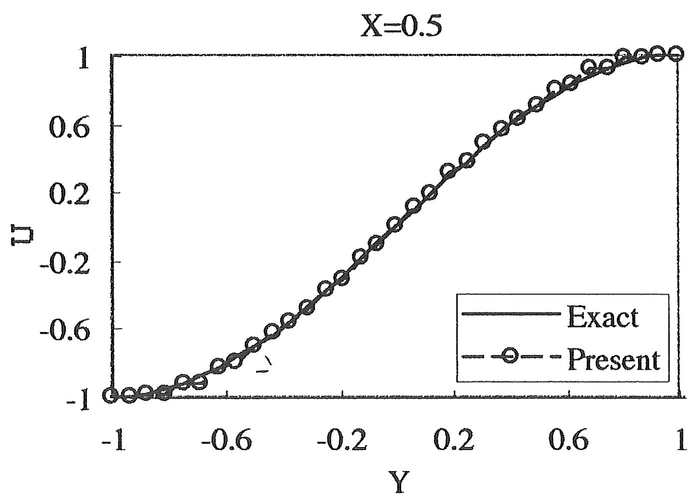
X	EXACT	RESULTS
-1	0.098	0.1075
-0.5	0.098	0.1146
0	0.098	0.1218
0.5	0.098	0.1146
1	0.098	0.1075



(a)

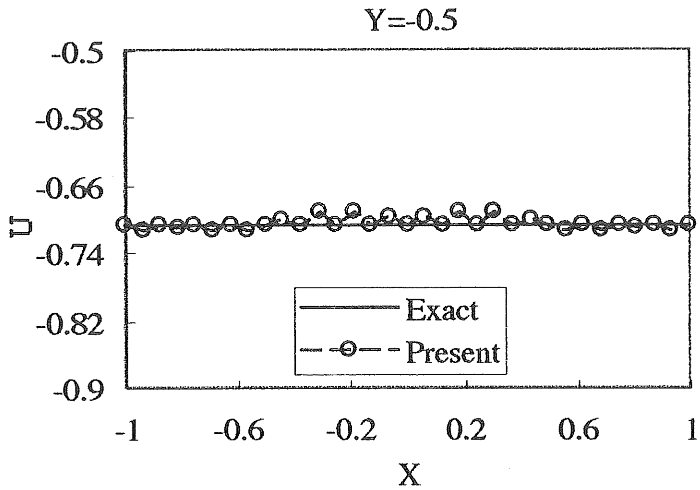


(b)

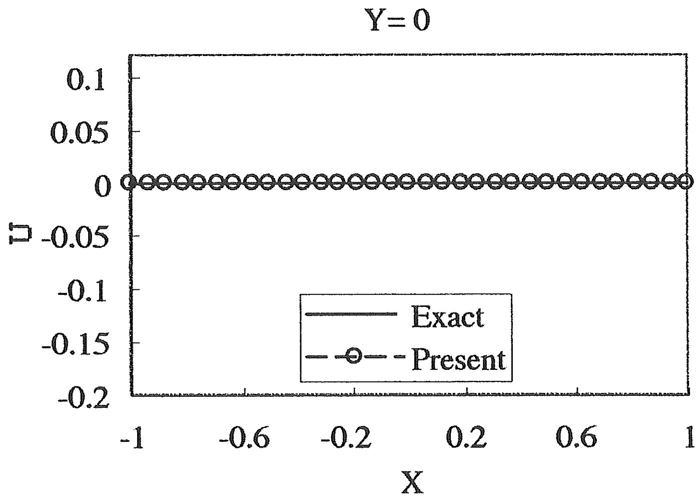


(c)

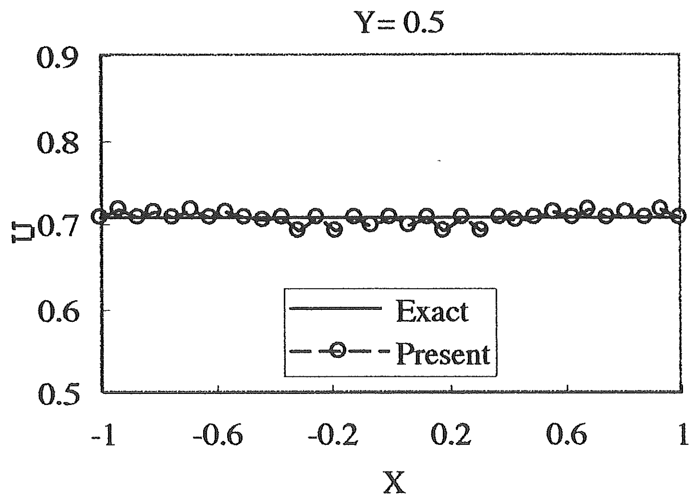
Figure 3.4: The numerical results of the the functional approximation when $f(x,y) = \sin(\frac{\pi y}{2})$ and utilizing the point source.



(d)

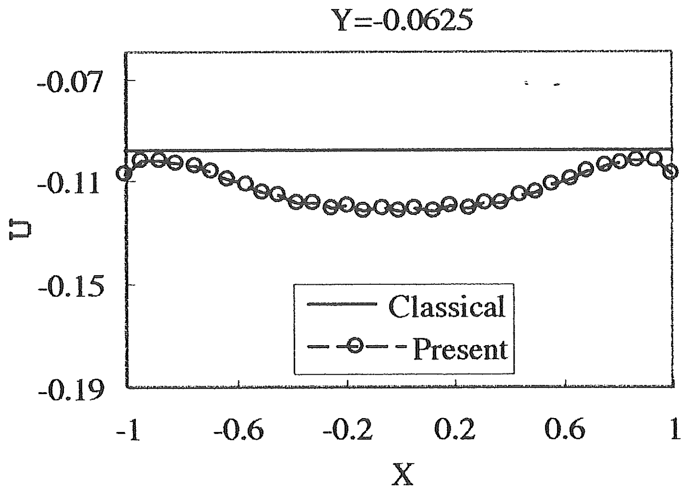


(e)

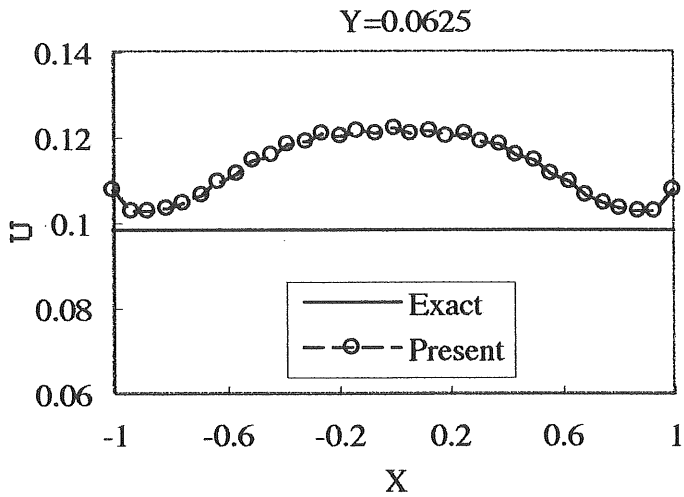


(f)

Figure 3.4: The numerical results of the functional approximation when $f(x,y)=\sin(\frac{\pi y}{2})$ and utilizing the point source.



(g)



(h)

Figure 3.4: The numerical results of the functional approximation

when $f(x,y)=\sin(\frac{\pi y}{2})$ and utilizing the point source.

Example 2

The functional approximation utilizing the line source is used for the same problem as the example 1. The line source is shown in Fig. 3.2. The numerical results are shown in table 3.2 and Fig. 3.5. Table 3.2 and Fig. 3.5 show that results by the present method

Table 3.2 (a): The functional approximation

when $f(x, y) = \sin(\frac{\pi y}{2})$ and utilizing the line source

X	Y	EXACT	RESULTS
0.5	-0.5	-0.7071068	-0.7071068
0.5	0.5	0.7071068	0.7071068
0	0	0	-2.77E-16
-0.5	-0.5	-0.7071068	-0.7071068
-0.5	0.5	0.7071068	0.7171068

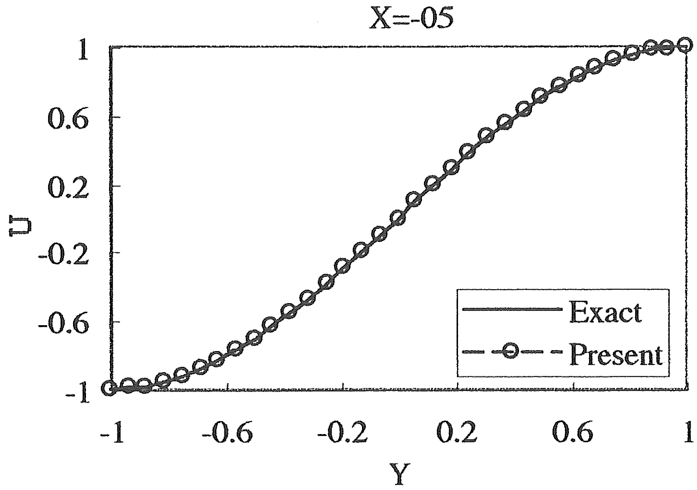
agrees well with that exact solution. It is clear that the discrete integral method utilizing the line source is superior to the discrete integral method utilizing the point source in the two-dimensional problem.

Table 3.2 (b): The functional approximation

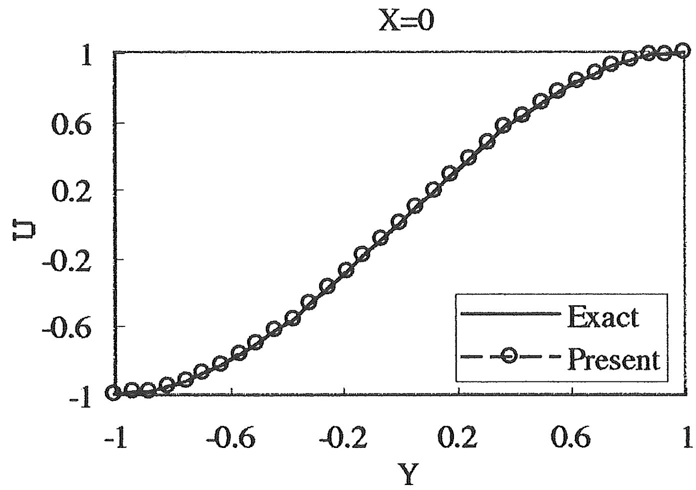
when $f(x, y) = \sin(\frac{\pi y}{2})$ and utilizing the line source

Y=0.0625

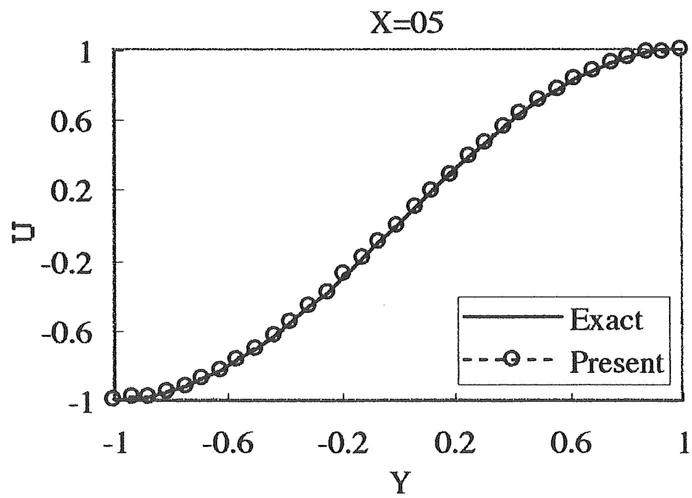
X	EXACT	RESULTS
-1	0.098	0.0975
-0.5	0.098	0.0975
0	0.098	0.0975
0.5	0.098	0.0975
1	0.098	0.0975



(a)

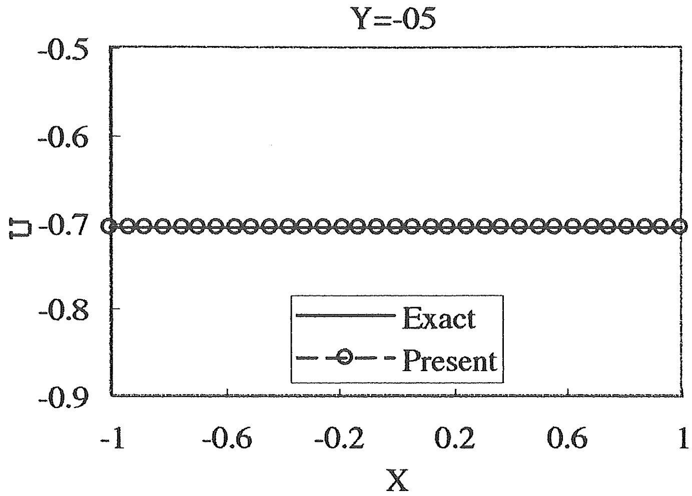


(b)

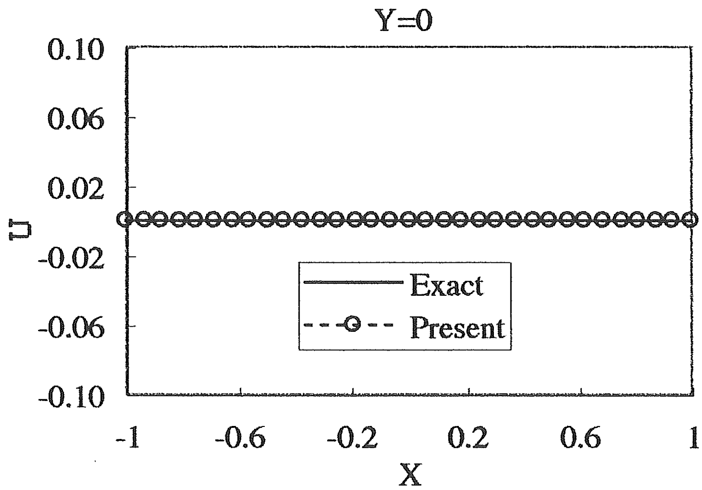


(c)

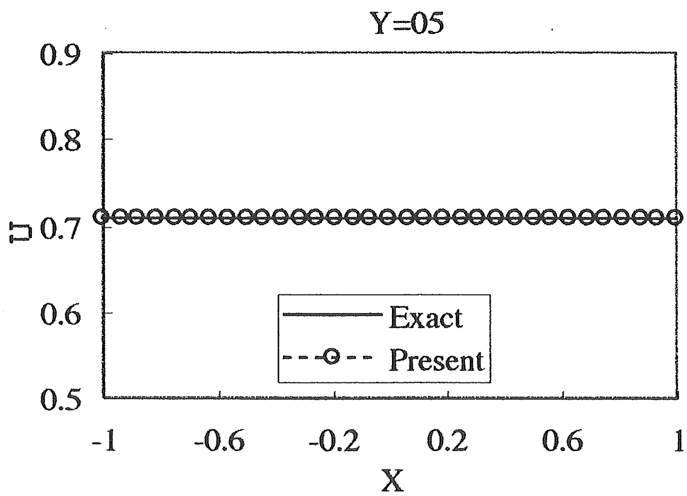
Figure 3.5: The functional approximation when $f(x, y) = \sin(\frac{\pi y}{2})$ and utilizing the line source



(d)



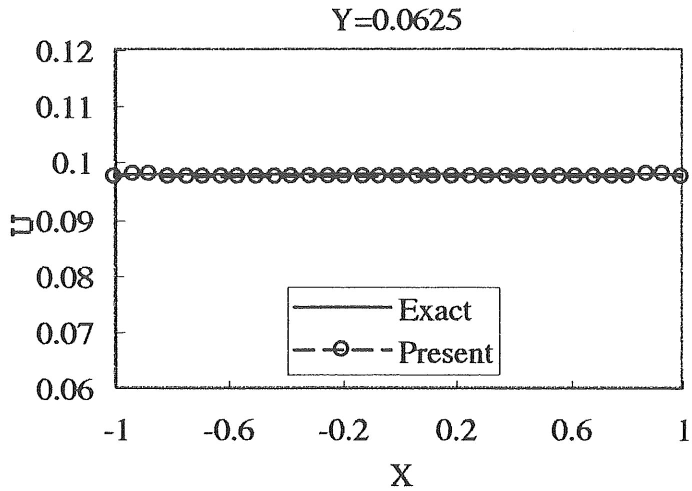
(e)



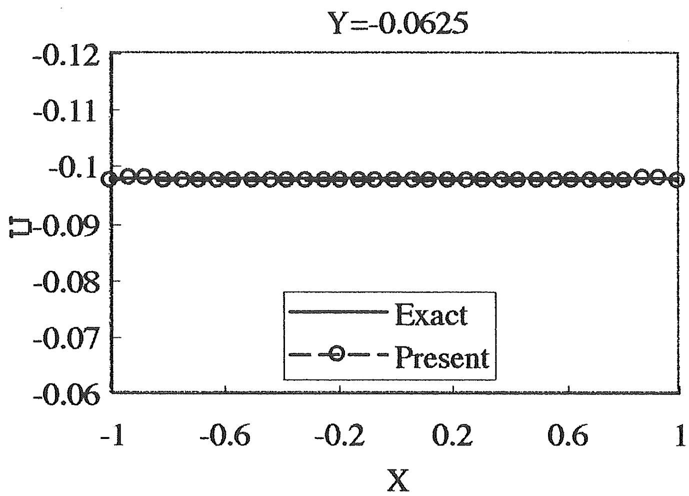
(f)

Figure 3.5: The functional approximation

when $f(x, y) = \sin(\frac{\pi y}{2})$ and utilizing the line source



(g)



(h)

Figure 3.5: The functional approximation

when $f(x, y) = \sin(\frac{\pi y}{2})$ and utilizing the line source

3.4 DISCRETE INTEGRAL METHOD [3-10]~[3-14]

In order to avoid using the higher order fundamental solutions and the derivatives of the heat source, the discrete integral method is proposed to employ in the domain integral and the general integra. This approach will be described as follows.

3.4.1 Discrete integral method utilizing the point source

The potential u under the heat source f satisfies the Poisson equation. This equation is rewritten as.

$$\nabla^2 u+\frac{1}{k}f=0 \tag{3.24}$$

where k is the thermal conductivity. The boundary integral equation for the potential in case of the problem is given by

$$kC(p)u(p)+\int_\Gamma q^*(Q,p)u(Q)d\Gamma(Q)$$

$$=\int_\Gamma u^*(Q,p)q(Q)d\Gamma(Q)+\int_\Omega f(Q)u^*(Q,p)d\Omega(Q) \tag{3.24}$$

As shown in Eq. (3.24), when there exits the heat source in the domain, the domain integral becomes necessary. Therefore, the discrete integral method utilizing the point source is applied to the domain integral. The first, a new function Z^* is introduced and it is defined as follows

$$\nabla^4 Z^*=u^*(Q,p) \tag{3.25}$$

where $u^*(Q,p)=\frac{1}{2\pi}\ln\left(\frac{1}{r}\right)$.

The second, using the functional approximation $(n=4)$ the heat source f is approximated by the following equation

$$\nabla^4 f=\sum_{i=1}^{m}B_i\delta(x-x_i) \tag{3.26}$$

where ∇^4 is duplicated harmonic operator, B_i is strength of the point source, x_i is position of the point source and m is number of the point source.

In terms of Eqs (3.25) and (3.26), the last term Eq. (3.24) can be written as

$$\int_{\Omega} f\,(Q)u^*(Q,p)d\Omega\,(Q) = \int_{\Omega} f\nabla^4 Z^* d\Omega$$

$$= \int_{\Gamma} (f\nabla^3 Z^* - \nabla f\,\nabla^2 Z^* + \nabla^2 f\nabla Z^* - \nabla^3 f Z^*)n\,d\Gamma$$

$$+ \sum_{i=1}^{m} B_i Z^*(x_i) \tag{3.27}$$

where the expression of Z^* can be evaluated as

$$Z^* = \frac{r^4}{256\pi}(3-2\ln r) \tag{3.28}$$

From the Eq. (3.27), the domain integral is evaluated by the boundary integral and the sum of the strength of the point source B_i and the function Z^* at the internal points.

In the following, another simpler approximation is described. If the function Z^* is introduced as follows

$$\nabla^2 Z^* = u^*(Q,\, p) \tag{3.29}$$

where $u^*(Q,\, p) = \frac{1}{2\pi}\ln\left(\frac{1}{r}\right)$

and the heat source is approximated as

$$\nabla^2 f = \sum_{i=1}^{m} D_i \delta(x - x_i) \tag{3.30}$$

the last term Eq. (3.24) can be derived as

$$\int_{\Omega} f\,(Q)u^*(Q,p)d\Omega\,(Q) = \int_{\Omega} f\nabla^2 Z^* d\Omega$$

$$= \int_{\Gamma} (\nabla Z^* f - Z^* \nabla f)\, n d\Gamma + \int_{\Omega} \nabla^2 f Z^* d\Omega$$

$$=\int_{\Gamma}(\nabla Z^{*}f\text{-}Z^{*}\nabla f)\,nd\Gamma+\int_{\Omega}\sum_{i=1}^{m}D_{i}\delta(x\text{-}x_{i})Z^{*}d\Omega$$

$$=\int_{\Gamma}(\nabla Z^{*}f\text{-}Z^{*}\nabla f)\,nd\Gamma+\sum_{i=1}^{m}Z^{*}(x_{i})D_{i} \tag{2.31}$$

where the expression of Z^{*} can be obtained as

$$Z^{*}(Q,p)=\frac{r^{2}}{8\pi}\left\{\ln\left(\frac{1}{r}\right)+1\right\} \tag{3.32}$$

As found in the above, the domain integral in the Poisson equation can be performed by the discrete integral method. In the following, general integral can also be evaluated by the discrete integral method. The integral is given as.

$$\int_{\Omega}fKd\Omega \tag{3.33}$$

where K is known function, f is a function which is either unknown or known. The function f is approximated using the point source as

$$\nabla^{n}f=\sum_{i=1}^{m}E_{i}\delta(x\text{-}x_{i}) \tag{3.34}$$

Further, a function Z^{*} introduced into equation which is defined by the next equation:

$$\nabla^{n}Z^{*}=K \tag{3.35}$$

When $n=2$, integrating Eq. (3.33) by part and then substituting Eq. (3.34) into it, the equation is charged into the following form.

$$\int_{\Omega}fKd\Omega=\int_{\Omega}f\nabla^{2}Z^{*}d\Omega$$

$$=\int_{\Gamma}(\nabla Z^{*}f\text{-}Z^{*}\nabla f)\,nd\Gamma+\sum_{i=1}^{m}Z^{*}(x_{i})D_{i} \tag{3.36}$$

3.4.2 Discrete integral method utilizing the line source

Using the line source as shown in Fig. 3.1 the function f is approximated by the following equation

$$\nabla^2 f=\sum_{i=1}^{m} L(s_i)\delta^L(s_i) \tag{3.37}$$

In the case of $n=2$, The integral can be evaluated from Eq. (3.33)

$$\nabla^2 Z^*=K \tag{3.38}$$

$$\int_\Omega fKd\Omega=\int_\Omega f\nabla^2 Z^* d\Omega$$

$$=\int_\Gamma (\nabla Z^* f\text{-}Z^*\nabla f)\,\boldsymbol{n}d\Gamma+\int_\Omega \nabla^2 f Z^* d\Omega$$

$$=\int_\Gamma (\nabla Z^* f\text{-}Z^*\nabla f)\,\boldsymbol{n}d\Gamma+\int_\Omega \sum_{i=1}^{m} L(s_i)\delta^L(s_i)\; Z^* d\Omega \tag{3.39}$$

This approach is called the discrete integral method utilizing the line source.

From Eqs (3.27), (3.31), (3.36) and (3.39) the domain integral or the general integral is evaluated by the boundary integral and the sum of the values at the internal points. In BEM, the domain integral is carried out not using the internal cells.

3.4.3 Numerical example

The discrete integral method utilizing the line source is presented for the heat conduction equation with a heat source in two dimensions as shown in Fig. 3.6. For this study, the heat conduction equation can be described as

$$\nabla^2 T + b = 0 \qquad \text{in} \ \ \Omega \tag{3.40}$$

where T is temperature and b is a heat source over the domain Ω.

In this case, the boundary conditions are written as follows.

$$T = 0; \qquad y = -1 \tag{3.41}$$

$$T = 100; \qquad y = 1 \tag{3.42}$$

$$q = 0; \qquad x = -1 \tag{3.43}$$

$$q = 0; \qquad x = 1 \tag{3.44}$$

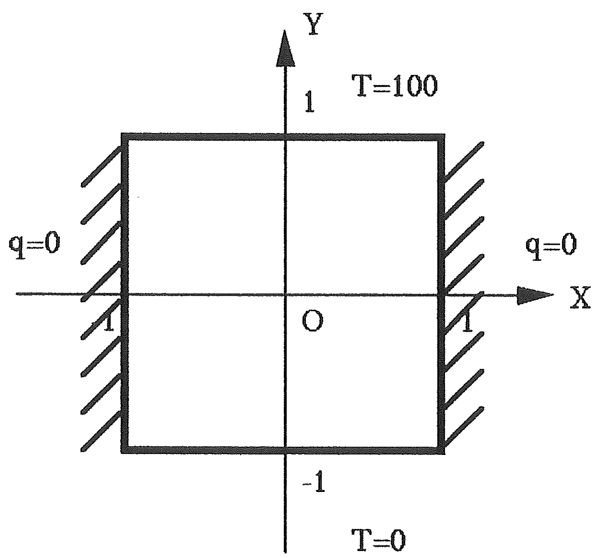
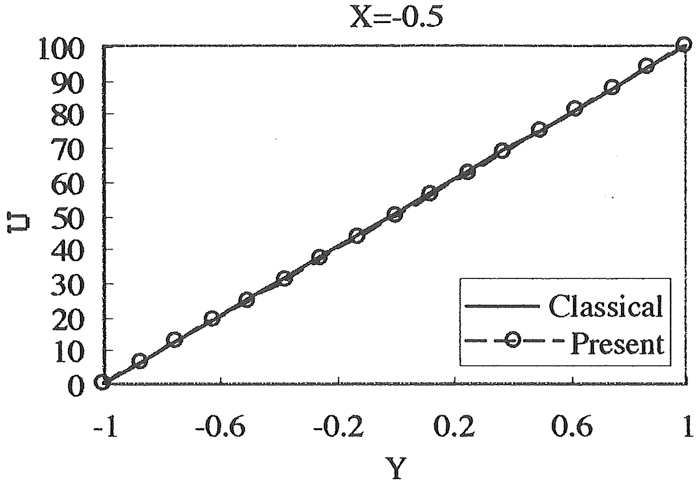


Figure 3.6: Two-dimesional heat conduction problem with a heat source

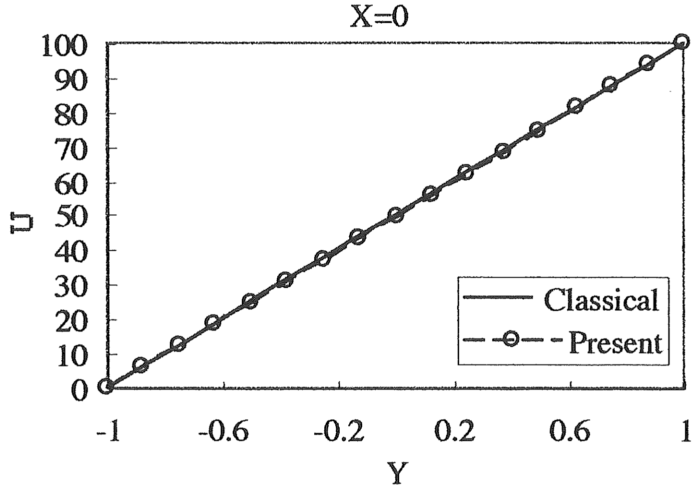
In this example, the heat source b is selected as $b(x, y) = -\cos(\pi x/2)\cos(\pi y/2)$ and the line sources are chosen as the form of Fig. 3.2, which is distributed evenly on the domain. There are 4 elements on the line source and the number of the line source is 17. The results are shown in Fig. 3.7 and table 3.3 and are compared with the classical BEM to show the correspondence of the solutions. The maximum error is only 1.8%.

Table 3.3: Temperature distribution when the heat source $b(x, y) = -\cos(\pi x/2)\cos(\pi y/2)$ and utilizing the line source

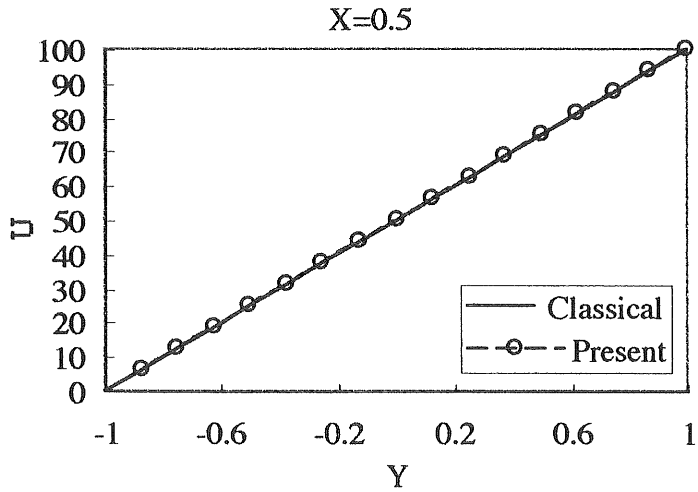
X	Y	EXACT	RESULTS
0.5	-0.5	25.1837	24.8015
0.5	0.5	75.1837	74.8015
0	0	50.2906	49.6943
-0.5	-0.5	25.1837	24.8015
-0.5	0.5	75.1837	74.8015



(a)

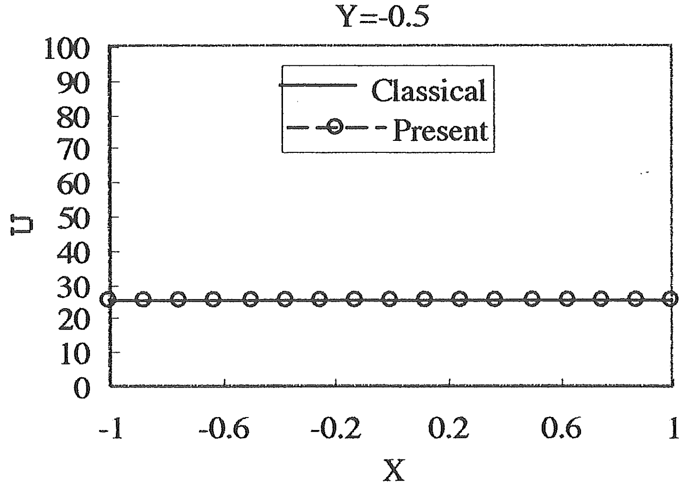


(b)

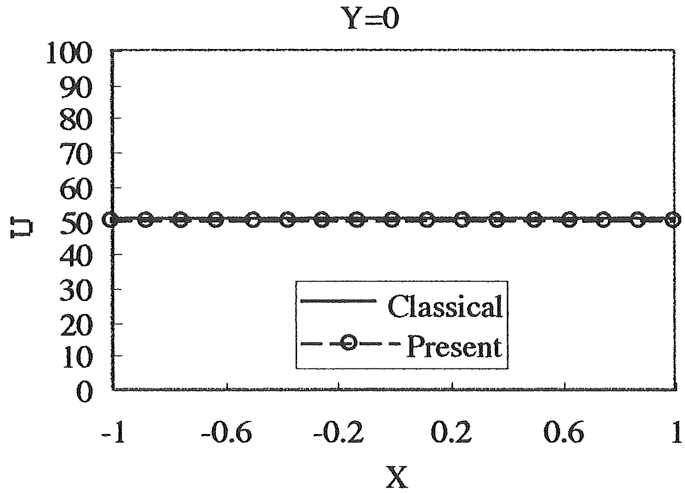


(c)

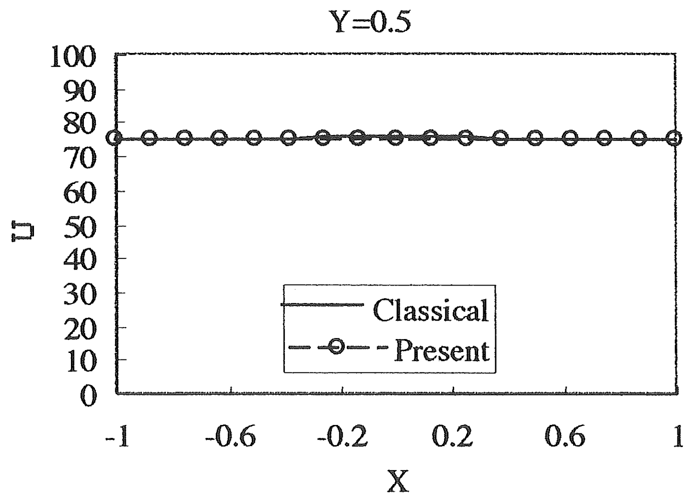
Figure 3.7: Temperature distribution when the heat source $b(x, y) = -\cos(\pi x/2)\cos(\pi y/2)$ and utilizing the line source



(d)



(e)



(f)

Figure 3.7: Temperature distribution when the heat source $b(x, y) = -\cos(\pi x/2)\cos(\pi y/2)$ and utilizing the line source

3.5 CONCLUSIONS

In this chapter, the basic theory of the functional approximation utilizing the point source and the line source is described. Using this functional approximation the discrete integral method is developed. Some conclusions are obtained as follows:

1. A new approach is used for the functional approximation utilizing the point source and the line source. The function is expressed by the boundary condition and the strength of the point source or the line source. By the known function this functional approximation is examined using the point source $(n = 2)$ or the line source. Using the line source, the numerical results agrees with that by the exact values in the two dimensions.

2. In the discrete integral method, the general integral can be expressed by the boundary integral and the sum of the values at the interior points. These formulas are derived using the point source $(n = 2,\ n = 4)$ and the line source.

3. The discrete integral method using the line source is applied to the heat conduction equation with a heat source in the two dimensions by BEM. Since the domain integral is evaluated by the boundary integral and the sum of the values at the interior points, the internal cells are not needed. The numerical results by this approach agree with that by the classical BEM. This method uses the lower order fundamental solution so that it is simpler than MRM.

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CHAPTER 4

A NEW ALGORITHM FOR BENDING PROBLEMS OF CONTINUOUS AND INHOMOGENEOUS BEAM BY THE BEM

4.1 INTRODUCTION

There are many occasions to deal with bending problems of the continuous beam in mechanical and structural design. Therefore, many solvers are developed for calculating them. It could be said that the boundary element method (BEM) [4-1], [4-2] has been satisfactorily established as one of the best solvers. Since total quantity of computation is not heavy in daily problems a personal computer is enough to perform such design works. It seems, in this sense, that the study again on reformulation of the BEM analysis method for the continuous beam has little meanings.

On the other hand, it has been found that the conventional algorithms are inefficient and remain many places to be improved. It is true that such defects may not become a serious problem as far as they are put to practical use for an individual calculation. However, once they are applied to a kind of optimal design with a certain optimization algorithm, for example, the genetic algorithm (GA) [4-3], things have an undesirable turn. Because great many repetitive calculations are required in such problems, small disadvantages in the conventional algorithms are amplified greatly and the cost of design work becomes very high.

From this point of view, it is intended to develop a new algorithm. Main points of this study are (1) to improve the composition of the simultaneous equations by introducing a new formulation process, (2) to establish a scheme without any variables at intermediate points; and (3) to establish a generalized solution scheme for an inhomogeneous beam. (4) to treat the domain integral by the discrete integral method utilizing the point source function. The domain integral is evaluated by the boundary integral and the sum of the values at some internal points. These new algorithms will reduce greatly the size of matrix as well as the computing time, and therefore, will bring about high efficiency on the repetitive calculations. As a result, it is realized to get the low cost of optimal design in daily work.

4.2 FORMULATION OF BEAMS BY BEM

At first, outline of the new algorithm developed in this study is described in contrast to the conventional formulation. They are summed in Table 4.1. The primary purpose of developing this new algorithm is to decrease the size of matrix and CPU time, bearing in mind that it may be applied to such as optimal problems which demand many iterating processes. The strategy it has been paid attention to is to simplify the simultaneous equations and to establish a scheme which allows to perform the calculation without values at any intermediate point (let us call it the nondividing scheme). The terms in Table 4.1 will be described in detail in the followings.

4.2.1 Basic equations for uni-span beam

As the system of simultaneous equations for uni-span beam, four equations should be supplied as mentioned later. In the conventional formulation [4-2], they consist of deflections and slopes (angles of deflection) at two ends of the beam. Contrary to this, it is noticed, in this study, that a simpler set of equations can be introduced. Namely, a set of each equation of deflection, slope, bending moment and shear force makes them up in the present formulation. Since the present formulation is much simpler, not only the computing gets quicker, but the programming for the nondividing algorithm becomes much more

Table 4.1: Comparison of the algorithm

Subject	Conventional	Present
Construction of four simultaneous equations	Eq.of $W_{p\to 0}$ & $_{p\to L.}$ Eq.of θ $_{p\to 0}$ & $_{p\to L}$	Eq.of $W_{p\to 0}$ & $_{p\to L.}$ Eq.of $\theta_{p\to 0}$ & $_{p\to L.}$ Eq. of $M_{p\to 0}$ & $_{p\to L.}$ Eq. of $Q_{p\to 0}$ & $_{p\to L}$
Beam with n intermediate simple support	Divide the beam into n+1 spans. Number of induced unknowns=4*n	Set n unknowns without division. Number of induced unknowns=n
Beam with p steps	Divide the beam into p+1 spans. Number of induced unknowns=4*p	No unknown induced
Beam subjected to m external moment loads	Divide the beam into m+1 spans. Number of induced unknowns=4*m	No unknown induced
General inhomogeous beam	Divide the bean into several steps or set the unknowns of W inside	Decide the weight function depending on the distribution of EI. No unknown induced
Domain integral	Divide the domain into cells	Obtain the boundary integral and the sum of the values at some points

simpler. Moreover, the present process is indispensable for formulation of the inhomogeneous beam.

4.2.1.1. Outline of conventional formulation

Let us review the procedure of conventional formulation [4-2] first. It goes without saying that uni-span problem can easily be solved by hand without exaggerated process here. It is just basic consideration needed for more complicated problems. Deflection of a beam like in Fig. 4.1 is governed by the following differential equation when E and I are constant:

$$EI \frac{d^4 W}{d\, x^4} - q(x) = 0 \tag{4.1}$$

where W is deflection, E is Young's modulus, I is second moment of area and q is external force acted on the beam. Eq. (4.1) is transformed starting from the following weighted residual form of

$$\int_0^L \left\{ EI \frac{d^4 W}{dx^4} - q(x) \right\} W^* dx = 0 \tag{4.2}$$

In the Eq. (4.2), L is length of the span and W^* is the weight function defined as a

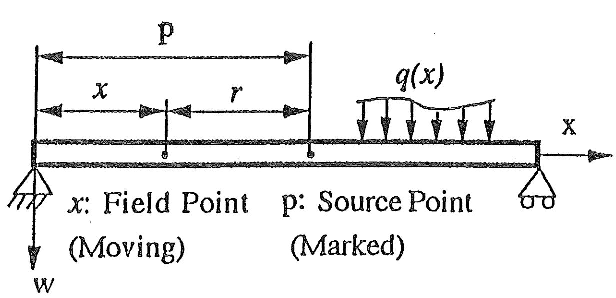

Figure 4.1: A uni-span beam

function of r, the distance between a source point p and a field point x which takes the form of

$$W^* = \frac{r^3}{12EI}, \quad (r=|x-p|) \tag{4.3}$$

After integrating Eq. (4.2) by part four times, the following equation is obtained as:

$$\left[EI \frac{d^3W}{dx^3} W^* - EI \frac{d^2W}{dx^2}\frac{dW^*}{dx} + EI \frac{dW}{dx}\frac{d^2W^*}{dx^2} - EI\, W \frac{d^3W^*}{dx^3} \right]_0^L$$

$$+ \int_0^L EIW \frac{d^4W^*}{dx^4} dx - \int_0^L q(x) W^*(x, p) = 0 \tag{4.4}$$

W^* satisfies the following equation in term of Eq. (4.3)

$$EI \frac{d^4W^*}{dx^4} = -\delta(x - p) \tag{4.5}$$

If Eq. (4.5) is substituted into Eq. (4.4), equation of deflection is got as

$$W(p) = EI \left[W \frac{d^3W*}{dx^3} - \theta \frac{d^2W*}{dx^2} - \frac{M}{EI}\frac{dW^*}{dx} + \frac{Q}{EI} W^* \right]_0^L$$

$$+ \int_0^L q(x)\, W^*(x, p) dx \tag{4.6}$$

Here, θ is slope, M is bending moment and Q is shear force, which are related to the derivatives of deflection as follows:

$$\frac{dW}{dx} = \theta(x), \quad \frac{d^2W}{dx^2} = -\frac{M(x)}{EI}, \quad \frac{d^3W}{dx^3} = -\frac{Q(x)}{EI} \tag{4.7}$$

It can be seen that Eq. (4.6) is described using eight values of W, θ, M and Q at both ends $(x=0,L)$. Four values among them are designated from boundary conditions at both ends but the rest are undetermined. So, four equations are needed to determine them. Two of them are given from Eq. (4.6) by letting $p=0$ and $p=L$. The rest of two can usually be obtained with the help of equation of slope. Actually, by differentiating Eq. (4.5) with respect to p, equation of slope is obtained as

$$\theta(p) = EI\left[-\theta\frac{d^2\widetilde{W^*}}{dx^2} - \frac{M}{EI}\frac{d\widetilde{W^*}}{dx} + \frac{Q}{EI}\widetilde{W^*}\right]_0^L + \int_0^L q(x)\,W^*(x,p)dx \qquad (4.8)$$

Here, the tilde mark on the variables denotes differentiation with respect to p. The same operations for letting $p=0$ and $p=L$ in the above yield two more equations. Then, they can be solved as a system of simultaneous equations of four unknowns. After that, deflection and slope at an interior point p can be calculated from Eqs (4.6) and (4.8), respectively.

By using the relationship in Eq. (4.7), equations of bending moment M and shear force Q at any point p are obtained:

$$M(p) = \left[-M\frac{d\widetilde{\widetilde{W^*}}}{dx} + Q\,\widetilde{\widetilde{W^*}}\right]_0^L + \int_0^L q(x)\,\widetilde{\widetilde{W^*}}(x,p)dx \qquad (4.9)$$

$$Q(p) = \left[Q\,\widetilde{\widetilde{\widetilde{W^*}}}\right]_0^L + \int_0^L q(x)\,\widetilde{\widetilde{\widetilde{W^*}}}(x,p)dx \qquad (4.10)$$

4.2.1.2 Reform of formulation process

Contrary to the usual procedure in the above, a simpler set of four equations is introduced following more concise formulation process. At first, the function is employed

$$W_Q^* = -\frac{1}{2}\mathrm{sgn}(x-p) \qquad (4.11)$$

as the weight function and consider the following weighted residual form of

$$\int_0^L \left(EI\frac{d^4W}{dx^4} - q(x)\right)W_Q^*dx = 0 \qquad (4.12)$$

Here, sgn is the sign function. When $x<p$, $\mathrm{sgn}(x-p) = -1$; When $x>p$, $\mathrm{sgn}(x-p) = 1$. Eq. (4.12) is executed integration by part once as follows:

$$\left[EI\frac{d^3W}{dx^3}W_Q^*\right]_0^L - \int_0^L EI\frac{d^3W}{dx^3}\frac{dW_Q^*}{dx}dx - \int_0^L q(x)W_Q^*(x,p)dx = 0 \qquad (4.13)$$

In term of Eq. (4.11), the following equation is obtained as

$$\frac{dW_Q^*}{dx} = -\delta(x - p) \tag{4.14}$$

It is noted that the second term on left-hand side of Eq. (4.13) is stated as

$$- \int_0^L EI \frac{d^3W}{dx^3}\frac{dW_Q^*}{dx}dx = \int_0^L EI \frac{d^3W}{dx^3}\delta(x - p)dx$$

$$= - Q(p) \tag{4.17}$$

Then using the result of Eq. (4.15), equation of shear force Q at p is obtained immediately as

$$Q(p) = \left[- QW_Q^*\right]_0^L - \int_0^L q(x)W_Q^*(x, p)dx \tag{4.16}$$

By letting $p \rightarrow 0$ and $p \rightarrow L$, the following equations are given as

$$Q(0) = \left[- QW_Q^*\right]_0^L - \int_0^L q(x)W_Q^*(x, 0)dx \tag{4.17}$$

$$Q(L) = \left[- QW_Q^*\right]_0^L - \int_0^L q(x)W_Q^*(x, L)dx \tag{4.18}$$

Similarly, by replacing the weighted residual form and the weight function in Eq. (4.12) with the following

$$\int_0^L \left(EI \frac{d^4W}{dx^4} - q(x)\right)W_M^*dx = 0, \quad W_M^* = -\frac{|x-p|}{2} \tag{4.19}$$

and

$$\int_0^L \left(EI \frac{d^4W}{dx^4} - q(x)\right)W_\theta^*dx = 0, \quad W_\theta^* = \frac{(|x-p|)^2}{4EI}\mathrm{sgn}(x - p) \tag{4.20}$$

From Eq. (4.19) integration by part twice, the following equation can be obtained

$$\left[EI\,\frac{d^3W}{dx^3}W_M^* \;-EI\,\frac{d^2W}{dx^2}\frac{dW_M^*}{dx}\right]_0^L + \int_0^L EI\,\frac{d^2W}{dx^2}\frac{d^2W_M^*}{dx^2}dx$$

$$-\int_0^L q(x)W_M^*(x,\,p)\,dx \tag{4.21}$$

Using Eq. (419), the following formula is stated as

$$\frac{d^2W_M^*}{dx^2} = -\delta(x-p) \tag{4.22}$$

Equation of bending moment M is

$$M(p) = \left[-M\frac{dW_M^*}{dx} + QW_{\mathrm{M}}^*\right]_0^L + \int_0^L q(x)W_M^*(x,\,p)\,dx \tag{4.23}$$

In the cases of $p \to 0$ and $p \to L$, Eq. (4.23) becomes

$$M(0) = \left[-M\frac{dW_M^*}{dx} + QW_{\mathrm{M}}^*\right]_0^L + \int_0^L q(x)W_M^*(x,\,0)\,dx \tag{4.24}$$

$$M(L) = \left[-M\frac{dW_M^*}{dx} + QW_{\mathrm{M}}^*\right]_0^L + \int_0^L q(x)W_M^*(x,\,L)\,dx \tag{4.25}$$

Eq. (4.20) is integrated by part three times

$$\left[EI\,\frac{d^3W}{dx^3}W_\theta^* - EI\,\frac{d^2W}{dx^2}\frac{dW_\theta^*}{dx} + EI\,\frac{dW}{dx}\frac{d^2W_\theta^*}{dx^2}\right]_0^L + \int_0^L EIW\,\frac{d^3W_\theta^*}{dx^3}dx$$

$$-\int_0^L q(x)W_M^*(x,\,p) = 0 \tag{4.26}$$

From Eq. (4.20) the following relation is derived as

$$EI\,\frac{d^3W_\theta^*}{dx^3} = -\delta(x-p) \tag{4.27}$$

equation of slope θ in term of Eq. (4.26) and Eq. (4.27) can be obtained

$$\theta(p) = EI\left[\theta\frac{d^2W_\theta^*}{d\,x^2} + \frac{M}{EI}\frac{dW_\theta^*}{dx} - \frac{Q}{EI}W_\theta^*\right]_0^L - \int_0^L q(x)W_\theta^*(x,\,p)\,dx \tag{4.28}$$

$\theta(0)$ and $\theta(L)$ satisfy equations as follows:

$$\theta(0) = EI\left[\theta\frac{d^2W_\theta^*}{d\,x^2} + \frac{M}{EI}\frac{dW_\theta^*}{dx} - \frac{Q}{EI}W_\theta^*\right]_0^L - \int_0^L q(x)W_\theta^*(x,0)\,dx \qquad (4.29)$$

$$\theta(L) = EI\left[\theta\frac{d^2W_\theta^*}{d\,x^2} + \frac{M}{EI}\frac{dW_\theta^*}{dx} - \frac{Q}{EI}W_\theta^*\right]_0^L - \int_0^L q(x)W_\theta^*(x,L)\,dx \qquad (4.30)$$

In the following, matrix form of this problem will described. In order to get the simple form, these variables are defined as

$$\frac{dW^*}{dx} = \theta^*(x),\quad \frac{d^2W^*}{dx^2} = -\frac{M^*(x)}{EI},\quad \frac{d^3W^*}{dx^3} = -\frac{Q^*(x)}{EI} \qquad (4.31)$$

Using these variables, deflection equations can be obtained at an internal point and both ends ($p = 0,\ L$) from Eq. (4.6).

$$W(p) = \left[Q(x)W_W^*(x,p) - M(x)\theta_W^*(x,p) + \theta(x)M_W^*(x,p) - W(x)Q_W^*(x,p)\right]_0^L$$
$$+\int_0^L q(x)W_W^*(x,p)dx \qquad (4.32)$$

$$W(0) = \left[Q(x)W_W^*(x,0) - M(x)\theta_W^*(x,0) + \theta(x)M_W^*(x,0) - W(x)Q_W^*(x,0)\right]_0^L$$
$$+\int_0^L q(x)W_W^*(x,0)dx \qquad (4.33)$$

$$W(L) = \left[Q(x)W_W^*(x,L) - M(x)\theta_W^*(x,L) + \theta(x)M_W^*(x,L) - W(x)Q_W^*(x,L)\right]_0^L$$
$$+\int_0^L q(x)W_W^*(x,L)dx \qquad (4.34)$$

The Slope equations can be rewritten in term of Eqs (4.28)~(4.30)

$$\theta(p) = \left[Q(x)W_\theta^*(x,p) - M(x)\theta_\theta^*(x,p) + \theta(x)M_\theta^*(x,p)\right]_0^L$$
$$+\int_0^L q(x)W_M^*(x,p)dx \qquad (4.35)$$

$$\theta(0) = \left[Q(x)W_\theta^*(x,0) - M(x)\theta_\theta^*(x,0) + \theta(x)M_\theta^*(x,0)\right]_0^L$$
$$+\int_0^L q(x)W_M^*(x,0)dx \qquad (4.36)$$

$$\theta(L) = \left[Q(x)W_{\theta}^{*}(x, L) - M(x)\theta_{\theta}^{*}(x, L) + \theta(x)M_{\theta}^{*}(x, L)\right]_{0}^{L}$$

$$+ \int_{0}^{L} q(x)W_{\theta}^{*}(x, L)dx \tag{4.37}$$

The bending moment equations and shear force equations are described as follows:

$$M(p) = \left[Q(x)W_{M}^{*}(x, p) - M(x)\theta_{M}^{*}(x, p)\right]_{0}^{L} + \int_{0}^{L} q(x)W_{M}^{*}(x, p)dx \tag{4.38}$$

$$M(0) = \left[Q(x)W_{M}^{*}(x, 0) - M(x)\theta_{M}^{*}(x, 0)\right]_{0}^{L} + \int_{0}^{L} q(x)W_{M}^{*}(x, 0)dx \tag{4.39}$$

$$M(L) = \left[Q(x)W_{M}^{*}(x, L) - M(x)\theta_{M}^{*}(x, L)\right]_{0}^{L} + \int_{0}^{L} q(x)W_{M}^{*}(x, L)dx \tag{4.40}$$

$$Q(p) = \left[Q(x)W_{Q}^{*}(x, p)\right]_{0}^{L} + \int_{0}^{L} q(x)W_{Q}^{*}(x, p)dx \tag{4.41}$$

$$Q(0) = \left[Q(x)W_{Q}^{*}(x, 0)\right]_{0}^{L} + \int_{0}^{L} q(x)W_{Q}^{*}(x, 0)dx \tag{4.42}$$

$$Q(L) = \left[Q(x)W_{Q}^{*}(x, L)\right]_{0}^{L} + \int_{0}^{L} q(x)W_{Q}^{*}(x, L)dx \tag{4.43}$$

(1) The value of boundary points

In the eight values of W, θ, M, Q at both ends, four values among them are given by boundary conditions at both ends and the other four values can be solved by the following matrix form.

$$HU+GT=B \tag{4.44}$$

Here, the matrices $[U]$, $[T]$ and $[B]$ are 1×4 and $[H]$, $[G]$ are 4×4 in size. Their forms are as follows.

$$U=\left[W(0)\theta(0)W(L)\theta(L)\right]^t \tag{4.45}$$

$$T=\left[M(0)Q(0)M(L)Q(L)\right]^t \tag{4.46}$$

$$H=\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1+M^*_\theta(0,0) & 0 & -M^*_\theta(L,0) \\ 1-Q^*_W(0,0) & M^*_W(0,0) & Q^*_W(L,0) & -M^*_W(L,0) \end{bmatrix} \tag{4.47}$$

$$G=\begin{bmatrix} 0 & 1+W^*_Q(0,0) & 0 & -W^*_Q(L,0) \\ 1-\theta^*_M(0,0) & W^*_M(0,0) & \theta^*_M(L,0) & -W^*_M(L,0) \\ -\theta^*_\theta(0,0) & W^*_\theta(0,0) & \theta^*_\theta(L,0) & -W^*_\theta(L,0) \\ -\theta^*_W(0,0) & W^*_W(0,0) & \theta^*_W(L,0) & -W^*_W(L,0) \end{bmatrix} \tag{4.48}$$

$$B=\begin{bmatrix} \int_0^L q(x)W^*_Q(x,0)dx \\ \int_0^L q(x)W^*_M(x,0)\,dx \\ \int_0^L q(x)W^*_\theta(x,0)dx \\ \int_0^L q(x)W^*_W(x,0)dx \end{bmatrix} \tag{4.49}$$

(2) The value of internal points

Using matrix equation (4.44), it can be seen that values of $W(0)$, $W(L)$, $\theta(0)$, $\theta(L)$,

$M(0)$, $M(L)$, $Q(0)$, $Q(L)$ are all known. So that values of an internal point of shear force Q bending moment M ,slop θ and deflection W can also immediately obtained.

$$Q(p)=-W_Q^*(0,p)Q(0)+W_Q^*(L,p)Q(L)+\int_0^L q(x)W_Q^*(x,p)dx \tag{4.50}$$

$$M(p)=-W_M^*(0,p)Q(0)+W_M^*(L,p)Q(L)+\theta_M^*(0,p)M(0)-\theta_M^*(L,p)M(L)$$

$$+\int_0^L q(x)W_Q^*(x,p)dx \tag{4.51}$$

$$\theta(p)=-W_\theta^*(0,p)Q(0)+W_\theta^*(L,p)Q(L)+\theta_\theta^*(0,p)M(0)-\theta_\theta^*(L,p)M(L)$$

$$-M_\theta^*(0,p)\theta(0)+M_\theta^*(L,p)\theta(L)+\int_0^L q(x)W_\theta^*(x,p)dx \tag{4.52}$$

$$W(p)=-W_W^*(0,p)Q(0)+W_W^*(L,p)Q(L)+\theta_W^*(0,p)M(0)-\theta_W^*(L,p)M(L)-M_W^*(0,p)\theta(0)$$

$$+M_W^*(L,p)\theta(L)+Q_W^*(0,p)W(0)-Q_W^*(L,p)W(L)+\int_0^L q(x)W_W^*(x,p)dx \tag{4.53}$$

Compose equations (4.50)~(4.53), the following equation can be obtained:

$$\begin{bmatrix} Q(p) \\ M(p) \\ \theta(p) \\ W(p) \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -M_\theta^*(0,p) & 0 & M_\theta^*(L,p) \\ Q_W^*(0,p) & -M_W^*(0,p) & -Q_W^*(L,p) & M_W^*(L,p) \end{bmatrix} U$$

$$+\begin{bmatrix} 0 & -W_Q^*(0,p) & 0 & W_Q^*(L,p) \\ \theta_M^*(0,p) & -W_M^*(0,p) & -\theta_M^*(L,p) & W_M^*(L,p) \\ \theta_\theta^*(0,p) & -W_\theta^*(0,p) & -\theta_\theta^*(L,p) & W_\theta^*(L,p) \\ \theta_W^*(0,p) & -W_W^*(0,p) & -\theta_W^*(L,p) & W_W^*(L,p) \end{bmatrix} T+\tilde{B} \tag{4.54}$$

Here, $\tilde{B}$ is an internal point integrating coefficient.

$$\tilde{B} = \begin{bmatrix} \int_0^L q(x) W_Q^*(x,p) dx \\ \int_0^L q(x) W_M^*(x,p)\, dx \\ \int_0^L q(x) W_\theta^*(x,p) dx \\ \int_0^L q(x) W_W^*(x,p) dx \end{bmatrix} \tag{4.55}$$

The equation of deflection is the same as the equation (4.6). In this chapter, a set of four equations are composed by letting either $p=0$ or $p=L$ in the equations (4.16), (4.23), (4.28) and, adding to them, equation (4.6). It is recognized that the present system of equations has become much more compact. In addition, the present process is based on the different concept from the conventional one. In other words, the weight functions in each equation are decided independently of others. It is no doubt that the final equations are the same although process is different. From this, the reform of concept may seem to be trivial. On the contrary, the reform is very important and it will be found that this process is indispensable in the following formulations, especially for the inhomogeneous beam.

4.2.2 Nondividing solution scheme

4.2.2.1 Treatment of a simply supporting point [4-5] [4-6]

Since reaction force at a simply supporting point becomes unknown, it can be not solved by the scheme for uni-span beam in the above section. In the conventional formulation, this problem is resolved by dividing the beam into two uni-span beams at that point. Each segment is regarded as a uni-span beam though excess unknowns appear on the two new ends. Actually, only $W=0$ is designated and the rest remains undetermined there. In this case, connective conditions of θ and M at this point are utilized to complement the lacking equations. The reaction force should be decided as the difference of shear forces on the two new ends. Thus, four unknowns, θ , M and two of Q, will increase at every supporting point. Then, it follows that a beam with n intermediate supporting points has the unknowns of $4*n+4$

In the present formulation, the nondividing scheme is developed. Namely, the reaction force at the supporting point is taken as an unknown external concentrated force and, corresponding to it, an equation of $W=0$ from equation (4.6) is complemented. This process can be carried out without adding any intermediate variables. The unknown reaction force is written from the term of external force, right hand side in equation (4.2) for example, as follows:

$$\int_0^L q(x)W^*(x,p)\,dx=\int_0^L R(x_0)\delta(x-x_0)W^*(x,p)dx$$

$$=R(x_0)W^*(x_0,p) \tag{4.56}$$

Where R is reaction force, x_0 is location of the supporting point and $\delta(x-x_0)$ is the delta function. In this formulation, a beam with n intermediate supporting points has the unknowns of just $n+4$.

If this manner is applied to the both ends, it brings the change of variables on two ends from shear force to reaction force. In other words, reaction force is unknown when the end is supported and it is zero when the end is free. Instead, shear force Q is zero at the both ends in any boundary conditions.

4.2.2.2 Concentrated moment load [4-7] [4-8]

In treating a concentrated moment load like in Fig. 4.2 by the conventional manner, the dividing scheme with the connective condition has also been applied. Then, four unknowns, W, θ, M and Q should be added at the acting point of every moment load. On the other hand, the nondividing scheme holds good also in this kind of load with the help of the operation of delta function [4-4] as follows.

In order to convert a concentrated moment load, M_0, into $q(x)$, distribution of moment $M(x)$ and shear force $Q(x)$ induced by M_0 should be determined. They are given as follows:

$$M(x) = M_0 \mathrm{sgn}(x-x_0) + \alpha x + \beta \tag{4.57}$$

$$Q(x) = \frac{dM}{dx} = M_0 \delta(x-x_0) + \alpha \tag{4.58}$$

Where x_0 is location of the moment load, α and β are constants related to boundary condition, etc. Then the equivalent external force, $q(x)$, are obtained as

$$q(x) = \frac{dQ}{dx} = M_0 \frac{d}{x} \delta(x - x_0) \tag{4.59}$$

According to the differentiation formula of delta function [4-4], the integration terms of external force are given a concrete form. For instance, the right side of equation (4.2) is written as

$$\int_0^L q(x) W^* dx = \int_0^L M_0 \frac{d}{dx} \delta(x - x_0) W^* dx$$

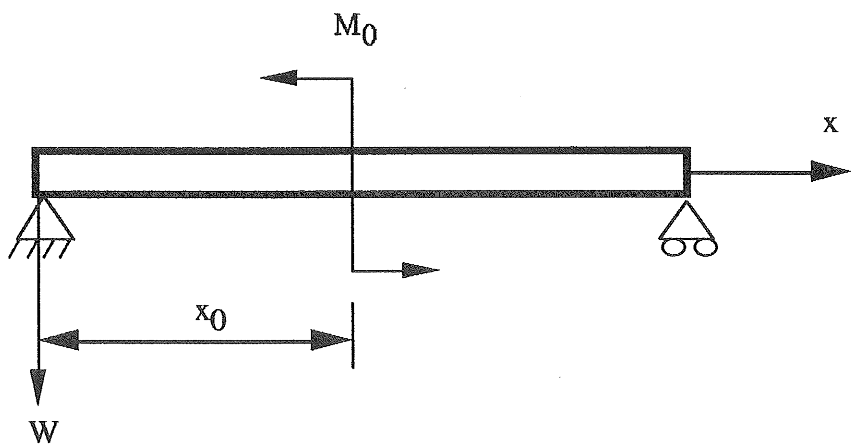

Figure 4.2: Concentrated monent load.

$$= \int_0^L -M_0\delta(x-x_0)\frac{dW^*}{dx}dx = -\frac{x_0^{\ 2}}{4EI}M_0\mathrm{sgn}(x_0 - p) \tag{4.60}$$

It should be noted that this manner does not yield any increase in additional unknowns.

4.2.2.3. A beam with n steps [4-6] [4-9]

In treating a beam with n steps as shown in Fig. 4.3, the usual formulation also requires the dividing scheme, in other words, additional four unknowns at the node between every two segments must exist. The nondividing scheme can be applied to this problem without adding any unknowns.

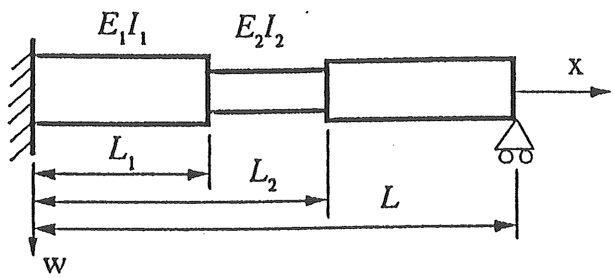

Figure 4.3: A beam with two steps

The weighted residual form are written, in this problem, with the weight function W^* as

$$\int_0^L \left[\frac{d^2}{dx^2}\left(EI\,\frac{d^2W}{dx^2} - q(x)\right)\right]W^*\,dx = 0 \tag{4.61}$$

Since it is possible to change the weight function every segment, the above expression can be rewritten as follows:

$$\int_0^{L_1} E_1I_1\frac{d^4W}{dx^4}W^*{}_1dx + \int_{L_1}^{L_2} E_2I_2\frac{d^4W}{dx^4}W^*{}_2dx + \cdots$$

$$= \int_{0}^{L_1} q(x)\, W^*{}_1 dx + \int_{L_1}^{L_2} q(x) W^*{}_2 dx + \cdots \tag{4.62}$$

Where L_i is coordinate of ith stepped node $(1 \le i \le n)$ and W_i^* is the undetermined weight function for the ith segment. In a similar way with the uni-span beam, the weight function is taken independently. In order to derive the equations of shear force Q and moment M at an interior point p_i, the function W_Q^* of equation (4.11) and W_M^* of equation (4.19) are employed as the weight functions. Since they are common through all the segments, all the intermediate values $Q(L_i)$ and $M(L_i)$ are cancelled. Finally, they can be expressed in the matrix form as follows:

$$Q(p_i) = [N]\begin{pmatrix} Q(0) \\ Q(L) \end{pmatrix} - \int_0^L q(x) W_Q^*(x, p_i) dx, \quad (L_{i-1} \le p_i \le L_i) \tag{4.63}$$

$$M(p_i) = [K]\begin{pmatrix} Q(0) \\ Q(L) \end{pmatrix} + [L]\begin{pmatrix} M(0) \\ M(L) \end{pmatrix} + \int_0^L q(x) W_M^*(x, p_i) dx, \quad (L_{i-1} \le p_i \le L_i) \tag{4.64}$$

Where the matrices $[N]$, $[K]$ and $[L]$ are coefficients related to the weight functions and are 1×2 in size.

If the weight function W_i^* is replaced by the function $W_{\theta i}^*$ of equation (4.20) with $EI = E_i I_i$ the equation of slope θ at an interior point p_i is obtained as follows:

$$\theta(p_i) = [A]\begin{pmatrix} \theta(0) \\ \theta(L) \end{pmatrix} + [B]\begin{vmatrix} M(L_1) \\ M(L_2) \\ \cdot \\ \cdot \\ \cdot \\ M(L_{n-1}) \end{vmatrix} + [C]\begin{vmatrix} Q(L_1) \\ Q(L_2) \\ \cdot \\ \cdot \\ \cdot \\ Q(L_{n-1}) \end{vmatrix}$$

$$+ [D]\begin{pmatrix} M(0) \\ M(L) \end{pmatrix} + [E]\begin{pmatrix} Q(0) \\ Q(L) \end{pmatrix} - \int_0^{L_1} q(x) W_{\theta 1}^*(x, p_i) dx$$

$$- \int_{L_1}^{L_2} q(x) W_{\theta_2}^*(x, p_i) dx - \cdots \tag{4.65}$$

Here, the matrices $[A]$, $[D]$ and $[E]$ are 1×2 and $[B]$ and $[C]$ are $1\times n$ in size. As found in the above, the intermediate values of $\theta(L_i)$ do not appear due to cancellation while $M(L_i)$ and $Q(L_i)$ are not cancelled automatically . However, they can be easily removed from the above using equations (4.65) and (4.66) by letting $p_i \rightarrow L_i$. After all, all the intermediate values are deleted and equation (4.67) can be finally described only by values at the two ends as follows:

$$\theta(p_i) = [A]\begin{pmatrix} \theta(0) \\ \theta(L) \end{pmatrix} + [M(0)M(L)](\hat{D}) + [Q(0)Q(L)](\hat{E}) + [P] \tag{4.66}$$

The hut mark denotes that some alterations are given and [P] denotes integral terms. The equations of deflection are obtained in a entirely similar way. Namely, the intermediate values of $W(L_i)$ and $\theta(L_i)$ are cancelled automatically and $M(L_i)$ and $Q(L_i)$ are deleted by using equations (4.63) and (4.64). Final form of the equation can be written using similar coefficient matrices as follows:

$$W(p_i) = [F]\begin{pmatrix} W(0) \\ W(L) \end{pmatrix} + [G]\begin{pmatrix} \theta(0) \\ \theta(L) \end{pmatrix} + [H]\begin{pmatrix} M(0) \\ M(L) \end{pmatrix} + [I]$$

$$\times\begin{pmatrix} Q(0) \\ Q(L) \end{pmatrix} + [J] \tag{4.67}$$

Thus, a system of simultaneous equations can be composed by letting $p_i = 0$ in equations (4.63), (4.64), (4.66) and (4.67).

4.2.3 Formulation for an Inhomogeneous Beam [4-10] [4-11]

In treating a beam with changeful rigidity (inhomogeneous beam including the non-prismatic) following conventional formulation, a special fundamental solution corresponding to a specific problem should be used [4-2]. Otherwise, the beam, in general problems other than the specific problem, has mainly been treated as the stepped beam divided into several segments. In this study, a general technique with the nondividing scheme has been developed. This scheme copes with the discontinuous change in bending rigidity by changing the fundamental solution every segment according to its state of change.

The weighted residual form for this problem can be written as

$$\int_0^L \left[\frac{d^2}{dx^2}\left(EI\,(x) \frac{d^2 W}{dx^2} \right) - q(x) \right] W^* dx = 0 \tag{4.68}$$

In this stage, the weight function is undetermined. In a similar way with the previous sections, the weight function is decided independently in every equation. For the equations of Q and M, it is possible to designate the functions of equations (4.11) and (4.19), respectively. Then, the identical equations with those for the uni-span beam of equations (4.16) and (4.23) can be obtained.

On the other hand, it is not self-evident what function is to be employed in order to derive the equations of θ and W. However that may be, let us integrate the above by part three times by letting $W^* = W_\theta^*$. Then it becomes

$$\int_0^L \frac{d^2}{dx^2}\left(EI\,(x) \frac{d^2 W}{dx^2} \right) W_\theta^*\, dx$$

$$= \left[\frac{d}{dx}\left(EI\,(x) \frac{d^2 W}{dx^2} \right) W_\theta^* \right]_0^L - \left[EI\,(x) \frac{d^2 W}{dx^2} \frac{dW_\theta^*}{dx} \right]_0^L$$

$$+ \left[EI\,(x) \frac{dW}{dx} \frac{d^2 W_\theta^*}{d\,x^2} \right]_0^L - \int_0^L \frac{dW}{dx} \frac{d}{dx}\left(EI\,(x) \frac{d^2 W_\theta^*}{dx^2} \right) dx \tag{4.69}$$

From this equation, it is found that the function should be decided so that it satisfies the following relation

$$\frac{d}{dx}\left(EI\,(x) \frac{d^2 W_\theta^*}{dx^2} \right) = \delta(x - p) \tag{4.70}$$

or

$$EI\,(x) \frac{d^2 W_\theta^*}{dx^2} = \frac{1}{2} \mathrm{sgn}(x - p) \tag{4.71}$$

Procedure hereafter has no choice but depends on the state of change in the bending rigidity $EI(x)$. In other words, the weight function should be changed depending on the concrete form of $EI(x)$. For example, if the function is given by a linear function as $EI\ (x)=E_0 I_0 (bx+c)$, the weight function is decided as

$$W_\theta^*(x, p) = \frac{\mathrm{sgn}(x\ \text{-}p)}{2bE_0 I_0}$$

$$\times\left\{\left(\frac{c}{b}+x\right)(\ln|bx+c| - \ln|bp+c|)-(x-p)\right\} \qquad (4.72)$$

Then, a compact expression of θ similar with the previous one has been successfully reached as follows:

$$\theta(p)=\left[EI\ (x)\theta\frac{d^2 W_\theta^*}{dx^2} + M\frac{dW_\theta^*}{dx} - QW_\theta^*\right]_0^L$$

$$-\int_0^L q(x) W_\theta^*(x, p)dx \qquad (4.73)$$

It should be noted that the weight functions for moment and shear force can never be produced from the weight function of equation (4.72).

Similarly, the weight function W_W^* for deflection is selected to satisfy the following relation:

$$\frac{d^2}{dx^2}\left(EI\ (x)\frac{d^2 W_W^*}{dx^2}\right) = \delta(x-p) \qquad (4.74)$$

Then, the weight function W_W^* is decided as

$$W_W^*(x, p)=\frac{\mathrm{sgn}(x-p)}{2bE_0 I_0}\left[\frac{(x-p)^2}{2}-\left(\frac{c}{b}+p\right)\right.$$

$$\left.\times\left\{\left(\frac{c}{b}+x\right)(\ln|bx+c| - \ln|bp+c|) - (x-p)\right\}\right] \qquad (4.75)$$

The equation of W is given by the following compact equation:

$$W(p) = \Bigg[EI\,(x) W \frac{d^3 W_W^*}{dx^3} - EI\,(x) \theta \frac{d^2 W_W^*}{dx^2} - M \frac{dW_W^*}{dx}$$

$$+ Q W_W^* \Bigg]_0^L + \int_0^L q(x) W_W^*(x, p) dx \tag{4.76}$$

The function of (4.75) also has no relation to the other weight functions of W_θ^*, W_M^* and W_Q^*. Therefore, it can be said that equations (73) and (76) are owing to the present formulation process.

Besides the above linear case of EI, the similar way to reach the fundamental solution with a closed form is also valid since the equation (4.71) can integrated analytically and so on as far as EI is expressed by a polynomial function including quadratic and cubic function. Moreover, there is no serious problem in this formulation scheme even if the state of change cannot be expressed by a sole function nor by an integral function. Because the weight function can be applied to only an assigned segment, several integral functions can be connected such that they fit the given function of the rigidity. Small errors, possibly included in certain circumstances, are to be negligible. From this fact, it may be enough to prepare several weight functions at most as occasion demands. In any case, the intermediate terms of $W(L_i)$ and $\theta(L_i)$ are cancelled automatically while the terms of $M(L_i)$ and $Q(L_i)$ are deleted by simple substitution, which is similar with the stepped beam.

4.2.4 Treatment of domain integral

In Eq. (4.18), (4.25), (4.30) and (4.34) the domain integral is written as follows:

$$\int_0^L q(x)\, W^*(x, p) dx \tag{4.77}$$

where $q(x)$ is the external force and W^* expresses W_Q^* , W_M^*, W_θ^* or W_W^*. In term of Eq.

(3.28) the external force is approximated utilizing the point source $(n = 2)$ as follows:

$$\nabla^2 q(x) = \sum_{i=1}^{m} r_i \delta(x - x_i) \tag{4.78}$$

The function Z^* is introduced as follows:

$$\nabla^2 Z^* = W^* \tag{4.79}$$

The next step, the domain integral of Eq. (4.77) can be transformed into the following form in term of Eqs. (4.79) and (4.78)

$$\int_0^L q(x) W^* dx = \int_0^L q(x) \nabla^2 Z^* dx$$

$$= \left(\frac{dZ^*}{dx} q(x) - Z^* \frac{dq}{dx} \right) \Bigg|_0^L + \int_0^L \nabla^2 q(x) Z^* dx$$

$$= \left(\frac{dZ^*}{dx} q(x) - Z^* \frac{dq}{dx} \right) \Bigg|_0^L + \sum_{i=1}^{m} Z^*(x_i)\, r_i \tag{4.80}$$

In order to identify the strength of the delta function r_i, the following equation can be obtained from Eq. (4.78)

$$\sum_{i=1}^{m} r_i Y^*(x_i, s) = \left(Y^* \frac{dq}{dx} - \frac{dY^*}{dx} q(x) \right) \Bigg|_0^L - c(s) q(s) \tag{4.81}$$

where $\nabla^2 Y^* = -\delta(x - x_i)$.

4.3 VERIFICATION OF THE PRESENT ALGORITHM

Several analysis examples are submitted to verify the validity of a new algorithm. Bearing in mind that the computing system is constructed on a personal computer, the program is coded using the Visual Basic on the machine with Windows NT. It does not matter to an individual calculation practically, even though there is difference in the quality of algorithm. However, in computing an optimal problem in which many iterative

Table 4.2: Results of continuous beam in Fig 4.4

	M_0/PL	Q_0/P	θ_L*(E_0I_0/PL^2)	$R_{0.3L}/P$	$R_{0.5L}/P$	R_L/P	Time(s)
Conventional	-0.038169	1.1451	-0.00032754	12.408	-1.3095	-0.19652	161
Present	-0.038169	1.1451	-0.00032754	12.408	-1.3095	-0.19652	10

calculation is required, its quality becomes an important issue because lowering of efficiency and increasing in cost of such works become not to be disregarded. Thus, computing time is measured through 1000 times calculation.

4.3.1. Continuous beam with simply supporting points

The first example is a continuous beam with nine intermediate simply supporting points (divide into ten equal parts) subjected to several concentrated forces and one moment load as shown in Fig. 4.4. Results from both the conventional and the present computing systems as well as their computing time are listed in Table 4.2. Since both systems gave the same results the new algorithm is proved to be valid. Computing time is cut down in a rate of about 1/16 in the new system. In addition, reduction in numbers of unknowns is also profitable. Actually, it is 13 in the present system while it is 44 in the conventional system.

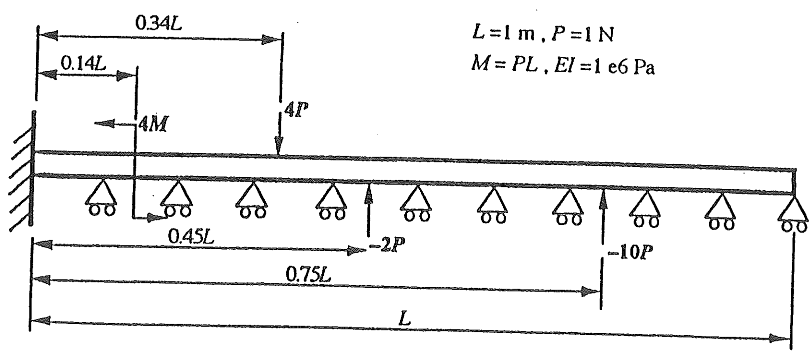

Figure 4.4: A continuous beam.

4.3.2 Inhomogeneous beam

An analysis example as shown in Fig. 4.5. is utilized in our verification of the new algorithm for an inhomogeneous beam. The bending rigidity is supposed to change according to the following linear function:

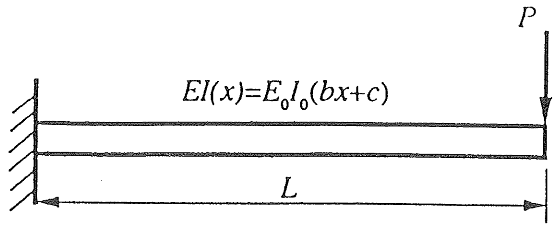

Figure 4.5: A nonprismatic cantilever

$$EI(x) = E_0 I_0 (bx + c) \tag{4.82}$$

Where $b = -0.5$ and $c = 1$ in this calculation. The conventional way which approximate the beam to a beam with three steps are tested in addition to the new algorithm. Their results and time for computation are shown in Table 4.3 compared with theoretical solutions, which are calculated by hand. The new algorithm is quite excellent in both of precision and speed. Contrary to this, it must be recognized that the conventional way of replacing by some steps gives inaccurate results.

Table 4.3: Results of non-prismatic beam in Fig. 4.5.

	M_0/PL	Q_0/P	$W_L*(E_0I_0/PL^3)$	$\theta_L*(E_0I_0/PL^2)$	Time(s)
Theoretical	1	1	0.38629	0.61371	
Conventional	1.0000	1.0000	0.39234	0.62048	12
Present	1.0000	1.0000	0.38629	0.61371	0.6

Table 4.4: Results of non-prismatic beam in Fig. 4.6

	M_0/PL	Q_0/P	$w_L*(E_0I_0/PL^3)$	$w_{L1}*(E_0I_0/PL^3)$	$\theta_L*(E_0I_0/PL^2)$	$\theta_{L1}*(E_0I_0/PL^2)$
Theoretical	1	1	2.7453	0.83333	2.1760	1.5
Present	1.0000	1.0000	2.7453	0.83333	2.1760	1.5000

The next example is a similar beam with changeful rigidity. The beam is composed of two segments, a constant part and a non-prismatic part as shown in Fig. 4.6.
The bending rigidity in the non-prismatic part changes with a linear function of equation (4.82) $(b = -2/3$ and $c = 5/3)$. It coincides with the constant part at the joint. As previously mentioned, it is needless to divide the beam into two pieces even in such a problem according to the present formulation. The results are listed in Table 4.4 together with the analytical solution calculated by hand. They testify the validity of the new algorithm.

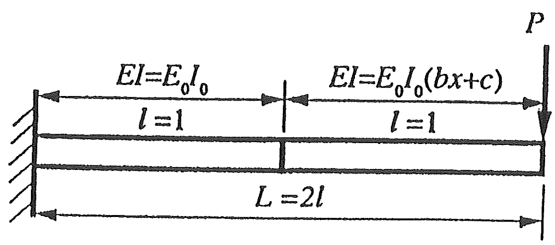

Figure 4.6: A cantilever with discontinuous EI.

Table 4.5: Results of Uni-span beam in Fig. 4.7

	$\theta(0)$	$\theta(L)$	$Q(0)$	$Q(L)$
Conventional	0.46389	-0.14722	0.91667	-1.75000
Present	0.46389	-0.14722	0.91667	-1.75000

4.3.3 Uni-span beam with an external force

In this example, uni-span beam with a external force is shown in Fig. 4.7 and the external force $q(x)$ is described as the following

$$q(x) = ax^2 + bx + c \tag{4.83}$$

where $a = 1$, $b = 1$ and $c = 2$. In this case, $L = 1$ and $EI = 1$.

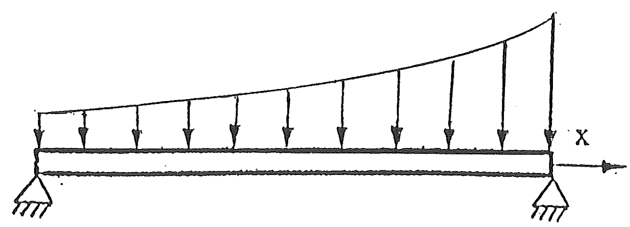

Figure 4.7: Uni-span beam with an external force.

The numerical results are compared with the classical BEM as shown in Table 4.5.

4.4 CONCLUSIONS

New algorithms without the dividing scheme have been established for bending problems of the continuous beam including the inhomogeneous beam by the BEM. This study is performed for the purpose of applying them to the practical optimal design works on a personal computer. Since the calculation for the optimal design by the GA, for instance, requires many repetitive calculation an algorithm with excellent quality is needed in order to reduce the cost of such works. The present algorithms produce some advantages of reducing the matrix size as well as computing time and of demanding much fewer resources, such as memory and capability. Therefore, it is natural to improve greatly the efficiency of numerously repetitive calculations in practical works. Main points of this study are summarized as follows:

1. Formulation process is improved to derive the system of simultaneous equations. As a result, they are reformed to be much more compact. This reform is of great benefit to configuring and simplifying the program based on the non-dividing scheme. This improvement is not a trivial rewriting but a indispensable notion for other problems like the inhomogeneous beam, etc.
2. The non-dividing scheme has been established contrary to the dividing scheme, which is needed in the conventional formulation on the occasions of simply supporting point, stepped beam, inhomogeneous beam (including discontinuously changeful rigidity) and concentrated moment load.
3. A general scheme for treating the inhomogeneous beam including those with discontinuously changeful rigidity has been established. This scheme is also based on the nondividing scheme.
4. The domain integral is treated by the discrete integral method utilizing the point source and is expressed by the boundary integral and the sum of the value at the some internal points. The interior cells can be not used.

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CHAPTER 5

ANALYSIS OF NONLINEAR HEAT CONDUCTION PROBLEMS

5.1 INTRODUCTION

The boundary element method is a well established numerical modelling technique for many types of linear problems in engineering and applied science. Recent development of BEM by numerous researchers [5-1]~[5-5] has demonstrated that BEM can also be used to solve nonlinear heat conduction problems with temperature dependence of thermal conductivity. In the following these results will be described.

C. A. Brebbia, S. Walker [5-1] and P. K. Banerjee, R. Butterfield [5-2] have first solved the steady-state nonlinear heat conduction equation with the temperature dependence of the thermal conductivity by the boundary element method. In this approach the corresponding differential equation is treated as the Poisson equation. Because the governing equation is nonlinear, the volume integral is included in the boundary integral equation. Although the volume integral can be computed by the classical BEM, the advantage of BEM is neglected. On the other hand, because temperature gradient is evaluated in the internal region, this approach not only demands additional calculations of the volume integral but also makes degree of accuracy decreased.

Yu. N. Akkuratov, V. N. Mikhailov [5-3] and R. Bialecki, A. Nowak[5-4] have applied Kirchhoff's transformation to the steady-state nonlinear heat conduction equation with the temperature dependence of the thermal conductivity independently. The governing equation is transformed into Laplace equation. This transformation transform the nonlinearity only into the boundary conditions. Therefore, it is unnecessary to establish the unknown quantity accompanied by the nonlinearity in the domain. The domain integral is not needed. However, since the inverse transformation is sometimes difficult, this method can not be generally used.

On the other hand, Kamiya, N. et.al [5-5] have proposed an approach by use of the new variable instead of the temperature. But it is limited to some special cases of the temperature dependence of the thermal conductivity.

As stated above, it is essential to seek for a new scheme, which avoids the disadvantages of Kichholff's transformation and the method use of the new variable. In this study, the discrete integral method [5-6] is attempted to apply to the nonlinear heat conduction problem by incorporating it into the boundary element method. The nonlinear equation is transformed into an equivalent form so that it may become possible to apply the discrete integral method. This approach is not limited by the temperature dependence of the thermal conductivity. In this thesis, the numerical analysis is carried out for two examples in one dimension and the computational results are compared with the exact solution. The nonlinear heat conduction problems with the temperature dependence of the thermal conductivity in the two and three dimensions will be carried out hereafter.

5.2 OTHER METHODS FOR SOLVING HEAT CONDUCTION EQUATION

In this section two approaches to the nonlinear problems of heat conduction will be described. In the first method the linearization of nonlinear heat conduction governing equation can be accomplished by the Kirchhoff's transformation [5-4], The second technique [5-5] defines the thermal conductivity and the variable $v=(k+a)T$ as the new variables instead of the temperature. When the thermal conductivity is linear, exponential or power functions, the governing equations of the thermal conductivity and the variable v are linear.

5.2.1 A method using Kirchhoff's transformation

Consider a two-dimensional region Ω with boundary Γ . The governing equation for the steady-state heat conduction with the temperature dependence of the thermal conductivity is described as

$$\frac{\partial}{\partial x}\left\{k\frac{\partial T}{\partial x}\right\}+\frac{\partial}{\partial y}\left\{k\frac{\partial T}{\partial y}\right\}=0 \qquad (\text{in } \Omega) \qquad (5.1)$$

$$k=f(T) \qquad (\text{in } \Omega) \qquad (5.2)$$

where T is the temperature, k is the thermal conductivity and f is the function that expresses the temperature dependence of the thermal conductivity. The boundary conditions are given as follows:

Dirichlet' s condition: $$T=T_0 \qquad (\text{on } \Gamma_1) \qquad (5.3)$$

Neumann' s condition: $$k\frac{\partial T}{\partial n}=q_0 \qquad (\text{on } \Gamma_2) \qquad (5.4)$$

Robin' s condition: $$k\frac{\partial T}{\partial n}=-h(T-T_f) \qquad (\text{on } \Gamma_3) \qquad (5.5)$$

The governing Eq. (5.1) is nonlinear because of the temperature dependence of the thermal conductivity. However, it can be linearized using the Kirchhoff' s transformation

$$t=U(T)=\int_0^T k(T)dT \qquad (5.6)$$

The original nonlinear Eq. (5.1) is reduced to the following equation

$$\nabla^2 t=0 \qquad (5.7)$$

The boundary conditions for Eq. (5.7) are obtained

$$t=U(T_0) \qquad (\text{on } \Gamma_1) \qquad (5.8)$$

$$\frac{\partial t}{\partial n}=k\frac{\partial T}{\partial n}=q_0 \qquad (\text{on } \Gamma_2) \qquad (5.9)$$

$$\frac{\partial t}{\partial n} = -h\left(U^{-1}[t] - U^{-1}[t_f]\right) \qquad \text{(on } \Gamma_3\text{)} \qquad (5.10)$$

where $T_f = U^{-1}[t_f]$. The inverse transformation is expressed as follows:

$$T = U^{-1}[t] \qquad (5.11)$$

In this problem the nonlinearity of the heat conduction equation can be transformed to the nonlinear boundary conditions using Kirchhoff's transformation and the resulting governing equation is the Laplace equation. The Kirchhoff's transformation still remains the nonlinearity of a boundary condition of the third kind. The only nonlinear equation is the boundary condition of the third kind. The nonlinearity, however, only involves the boundary nodes that have nonlinear boundary conditions. The Newton-Raphson's method is used to solve the nonlinear matrix equation. However, it is not easy to obtain the analytical solution from Eq. (5.11).

5.2.2 A method by the use of the new variable

This method by use of the new variable is different from the conventional Kirchhoff's transformation. In this approach, the governing equations of the thermal conductivity and the variable v is liner. Because the thermal conductivity and the variable v are functions of the temperature T, if the thermal conductivity $k(T)$ and the variable v become the known function, the temperature T also becomes the known. This method is described as follows:

Eq. (5.1) is transformed into the following equation

$$\frac{\partial k}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial k}{\partial y}\frac{\partial T}{\partial y} + k\nabla^2 T = 0 \qquad (5.12)$$

When the thermal conductivity is described by the exponential or the power functions in terms of the temperature, the equations of the thermal conductivity are written as

$$k = a\exp(bT) \qquad (5.13)$$

$$k = ab^T \qquad (5.14)$$

where a and b are constants. Now, the thermal conductivity k is regarded as the new variable and the governing equation that the variable k satisfies is proposed.

By taking the Laplace operator of Eqs. (5.13) and (5.14) respectively, the following equations are given as

$$\nabla^2 k = b\left\{\frac{\partial k}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial k}{\partial y}\frac{\partial T}{\partial y} + k\,\nabla^2 T\right\} \tag{5.15}$$

$$\nabla^2 k = \left\{\frac{\partial k}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial k}{\partial y}\frac{\partial T}{\partial y} + k\nabla^2 T\right\}\log b \tag{5.16}$$

If Eq. (5.12) is substituted into Eqs. (5.15) and (5.16) respectively, Eq. (5.15) is equivalent to Eq. (5.16). This equation is written as

$$\nabla^2 k = 0 \tag{5.17}$$

The appropriate boundary conditions for the Eq. (5.15) are given

$$k = a\exp\left(bT_0\right) = k_0 \tag{5.18}$$

$$\frac{\partial k}{\partial n} = bq_0 = \left\{\frac{\partial k}{\partial n}\right\}_0 \tag{5.19}$$

$$\frac{\partial k}{\partial n} = -h\left\{\frac{1}{b}(\log k - \log a) - T_f\right\} \tag{5.20}$$

In the similar way, the boundary conditions of Eq. (5.16) are also obtained as

$$k = ab^{T_0} \tag{5.21}$$

$$\frac{\partial k}{\partial n} = q_0 \log b = \left\{\frac{\partial k}{\partial n}\right\} \tag{5.22}$$

$$\frac{\partial k}{\partial n} = -h\left\{\frac{1}{\log b}(\log k - \log a) - T_f\right\} \tag{5.23}$$

When the thermal conductivity $k(T)$ is described by a linear function in terms of the temperature, the equation of thermal conductivity is

$$k = a + bT \tag{5.24}$$

where a and b are constants. If a variable v is defined as $v = (k + a)T$, the governing equation is obtained

$$\nabla^2 v = 0 \tag{5.25}$$

The boundary conditions of the variable v are written as follows:

$$v = T_0\left[k(T_0) + a\right] = v_0 \tag{5.26}$$

$$\frac{\partial v}{\partial n} = 2q_0 = \left\{\frac{\partial v}{\partial n}\right\}_0 \tag{5.27}$$

$$\frac{\partial v}{\partial n} = -2h\left\{\frac{v}{k+a} - T_0\right\} = -2h\left\{\frac{v}{a+\sqrt{(a^2 + bv)}} - T_f\right\} \tag{5.28}$$

Because the functions k and v are used as the new variables, the governing equations are expressed by the Laplace equation with nonlinear boundary conditions. Although this method does not need the complicated inverse transformation, it is suitable to three cases of the thermal conductivity only.

5.3 TWO-DIMENSIONAL PROBLEM

In order to avoid the complex inverse transform using the Kirchhoff's transformation, and the limitation of three cases of the thermal conductivity using a method by the use of the new variable, the discrete integral method in the Chapter 3 can be used for steady-state nonlinear heat conduction equation [5-6]. In this thesis, the governing nonlinear heat conduction equation is transformed into a new form to which the discrete integral method is applicable. The new equation can be solved instead of the original equation. The volume integration is transformed into the boundary integral and the sum of the internal points. This new approach is not restricted to the temperature dependence of the thermal conductivity. Its validity through one-dimensional example will be described in section 5.5.

5.3.1 The new governing equation and its boundary conditions

The main objective of this stage is to obtain the governing equation from Eqs. (5.1) and (5.2). By taking the Laplace operator of Eq. (5.2), the first step leads to as follows:

$$\nabla^2 k = f''(T)\left(\frac{\partial T}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y}\frac{\partial T}{\partial y}\right) + f'(T)\left(\frac{\partial^2 T}{\partial^2 x} + \frac{\partial^2 T}{\partial^2 y}\right) \tag{5.29}$$

On the other hand, Eq. (5.1) is treated as

$$f'(T)\left(\frac{\partial T}{\partial x}\frac{\partial T}{\partial x} + \frac{\partial T}{\partial y}\frac{\partial T}{\partial y}\right) + k\nabla^2 T = 0 \tag{5.30}$$

Taking the Eqs. (5.29) and (5.30) into account when $f''(T) \neq 0$ and $f'(T) \neq 0$, the following equation can be obtained as

$$\nabla^2 k = \frac{f'(T)f'(T) - f(T)f''(T)}{f'(T)}\nabla^2 T = \Phi(T)\nabla^2 T \tag{5.31}$$

The Eq. (5.31) is the governing equation which we want to get. If the variable is described by the thermal conductivity, the boundary conditions for the Eq. (5.31) are expressed by the known function as follows:

Dirichlet' s condition: $$k = f(T_0) = k_0 \tag{5.32}$$

Neumann' s condition: $$\frac{\partial k}{\partial n} = \frac{f'(T)}{f(T)} q_0 \tag{5.33}$$

Robin' s condition: $$\frac{\partial k}{\partial n} = -h(T - T_f)\frac{f'(T)}{f(T)} \tag{5.34}$$

This governing Eq. (5.31) is nonlinear with the nonlinear boundary conditions (5.32)~(5.34). In the following the discussion of Eq. (5.31) will be turned to.

1. If $f'(T)=0$, $k(T)$=constant.

The governing Eq. (5.1) becomes the Laplace equation as

$$\nabla^2 T = 0 \tag{5.36}$$

The Laplace equation can be resolved by the classical BEM.

2. If $f''(T) = 0$, $k(T) = a + bT$.

Here, a and b are arbitrary contstants. In this case, the left-hand side of Eq. (5.31) can be rewitten as the following form

$$\nabla^2 k = \nabla^2 (a+bT) = b\nabla^2 T \tag{5.36}$$

On the other hand, the right-hand side of Eq. (5.31) can be expressed as

$$\Phi(T)\nabla^2 T = b\nabla^2 T \tag{5.37}$$

Therefor, Eq. (5.31) becomes the identical relation. In order to obtain the solution of Eq. (5.1), the another variable must be defined. In the thesis, this variable is the same as the v in Eq. (5.25). Its governing equation and boundary conditions are Eqs. (5.25)~(5.28).

3. If the temperature dependence of the thermal conductivity is exponential or power functions, $\Phi(T)=0$.

The Eq. (5.31) becomes

$$\nabla^2 k=0 \tag{5.38}$$

It is obvious that Eq. (5.31) can be employed in all sorts of the temperature dependence of the thermal conductivity. Eq. (5.31) and boundary conditions (5.32) ~ (5.34) are general forms.

5.3.2 The new solution using discrete integral method

If both sides of the Eq. (5.31) are multiplied by the fundamental solution $u^*(Q, p)$ and is integrated over the domain, the following equation can be obtained as

$$\int_\Omega \nabla^2 k(Q) u^*(Q, p) d\Omega(Q) = \int_\Omega \Phi(T(Q))\, \nabla^2 T(Q) u^*(Q, p) d\Omega(Q) \tag{5.39}$$

The left -hand side can be transformed into boundary integration as

$$c(p)k(p) + \int_\Gamma \left(u^*(Q, p)\frac{\partial k(Q)}{\partial n} + k(Q)\frac{\partial u^*(Q, p)}{\partial n} \right) d\Gamma \tag{5.40}$$

The function $\nabla^2 T$ of Eq. (5.40) is approximated using the discrete integral method. In the Chapter 3 Eq. (3.7) is given as

$$\nabla^n f = \sum_{i=1}^{m} r_i \delta(Q - p_i) \tag{5.41}$$

where r_i is the strength of delta function and m is the number of the source point. If Eq. (5.41) is substituted into Eq. (5.40) when n=2 and $f = T$, the right- hand side is changed as follows:

$$\int_\Omega \Phi(T(Q)) \nabla^2 T(Q)\, u^*(Q, p) d\Omega(Q) = \int_\Omega \Phi(T(Q)) \sum_{i=1}^{m} r_i \delta(Q - p_i) u^*(Q, p) d\Omega(Q)$$

$$= \sum_{i=1}^{m} \Phi(T(p_i))\, r_i u^*(p_i, p) \tag{5.42}$$

Eq. (5.31) can be written as

$$c(p)k(p) + \int_\Gamma \left(u^*(Q, p) \frac{\partial k(Q)}{\partial n} + k(Q) \frac{\partial u^*(Q, p)}{\partial n} \right) d\Gamma(Q)$$

$$= \sum_{i=1}^{m} \Phi(T(p_i))\, r_i u^*(p_i, p) \tag{5.43}$$

where p is the internal point or the boundary point. $c(p)$ is the position coefficient. If the point p is located on a boundary, $c(p)$ =1/2. If the point p is located inside the domain, then $c(p)$=1.

When n=2 and f=T , if Eq. (5.41) is multiplied by fundamental solution $u^*(Q,p)$ and is integrated in the domain, Eq. (5.41) can be written as

$$\int_\Omega \nabla^2 T(Q) u^*(Q, p) d\Omega(Q) = \int_\Omega \sum_{i=1}^{m} r_i \delta(Q - p_i) u^*(Q, p) d\Omega(Q) \tag{5.44}$$

By putting the integration terms of the right-hand side on the boundary, the equation is obtained:

$$c(p)T(p) + \int_{\Gamma}\left(u^*(Q, p)\frac{\partial T(Q)}{\partial n} + T(Q)\frac{\partial u^*(Q, p)}{\partial n}\right)d\Gamma(Q)$$

$$= \sum_{i=1}^{m} r_i u^*(p_i, p) \tag{5.45}$$

5.4 DISCRETIZATION OF THE BOUNDARY INTEGRATION

The main problem is how to solve the boundary integral Eqs. (5.43) and (5.45). Because it is almost impossible to obtain the analytical solution, we divide the boundary into many elements, then integrate each element and express the boundary integral equation with the sum of finite term of integrations. If the boundary is divided into NE elements and the domain is divided into M cells, the Eqs. (5.43) and (5.45) can be written as

$$\sum_{i=1}^{NE}\int_{\Gamma_i} u^*(Q, p)q(Q)\left(-\frac{f(T(Q))}{f(T(Q))}\right)d\Gamma_i(Q) - \sum_{i=1}^{NE}\int_{\Gamma_i} q^*(Q, p)f(T(Q))d\Gamma_i(Q) - c(p)f(T(p))$$

$$-\sum_{i=1}^{m}\Phi(T(p_i))r_i u^*(p_i, p) = 0 \tag{5.46}$$

$$\sum_{i=1}^{NE}\int_{\Gamma_i} u^*(Q, p)q(Q)\left(-\frac{1}{f(T(Q))}\right)d\Gamma_i(Q) - \sum_{i=1}^{NE}\int_{\Gamma_i} q^*(Q, p)T(Q)d\Gamma_i(Q) - c(p)T(p)$$

$$-\sum_{i=1}^{m} r_i u^*(p_i, p) = 0 \tag{5.47}$$

The simultaneous equations consist of the Eq. (5.46) and the Eq. (5.47), letting $p=p_1, \cdots, p=p_m$. The simultaneous equations can be solved by the Newton -Raphson method [5-7]~[5-11] though the equation is nonlinear with nonlinear boundary conditions.

5.5 ONE-DIMENSIONAL PROBLEM

Sometime, it is necessary to solve one-dimensional nonlinear heat conduction problem.

In this section, the one-dimensional problem as shown in Fig. 5.1 will be described. The governing equation and the boundary condition are written as

$$\frac{d}{dx}\left(k(T)\frac{dT}{dx}\right) = 0 \qquad a \le x \le b \tag{5.48}$$

$$k = f(T) \tag{5.49}$$

The boundary condition is given as

$$T_{x=a} = T_a \qquad T_{x=b} = T_b \tag{5.50}$$

In a similar way with the two-dimensional heat conduction problem, the following equation is obtained as

$$\frac{d^2k}{dx^2} = \Phi(T)\,\frac{d^2T}{dx^2} \tag{5.51}$$

where $\Phi(T) = \dfrac{f'(T)f'(T)-f(T)f''(T)}{f'(T)}$. From Eq. (5.51) the following equation is got as

$$\left[u^*\frac{dk}{dx} - k\frac{du^*}{dx}\right]_a^b - k(p) = \int_a^b \Phi(T)\frac{d^2T}{dx^2}u^*(x, p)dx \tag{5.52}$$

where $u^*(x,p) = -\frac{1}{2}|x-p|$. In term of the discrete integral method temperature is expressed as

$$\nabla^2 T = \sum_{i=1}^{m} r_i \delta(x-x_i) \tag{5.53}$$

Using Eq. (5.53) Eq. (5.52) becomes the following form

$$\left[u^*\frac{dk}{dx} - k\frac{du^*}{dx}\right]_a^b - k(p) = \sum_{i=1}^{m} \Phi(T_i)\, r_i\, u^*(x_i, p) \tag{5.54}$$

By Green's identity Eq. (5.53) is charged into the following equation

$$\left[u^*\frac{dT}{dx} - T\frac{du^*}{dx}\right]_a^b - T(p) = \sum_{i=1}^{m} r_i\, u^*(x_i, p) \tag{5.55}$$

The simultaneous equations consist of the Eq. (5.54) and the Eq. (5.55).

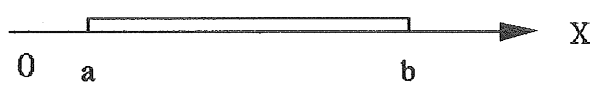

Figure: 5.1 One-dimensional problem

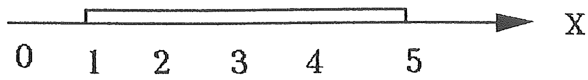

Figure: 5.2 Analysis model

5.6 NUMERICAL EXAMPLES

Although the two-dimensional formulas have been derived, here the only one-dimensional problems are solved. The nonlinear heat conduction problems in two and three dimensions will be carried out hereafter. Now as the first example, the one-dimensional problem is considered as shown in Fig. 5.2. The one-dimensional heat conduction equation is written as

$$\frac{d}{dx}\left(k(T)\frac{dT}{dx}\right)=0 \qquad 1\leq x\leq 5 \qquad (5.56)$$

In this case, the boundary conditions are expressed as

$$T_{x=1}=2 \qquad T_{x=5}=8 \qquad (5.57)$$

In this problem the temperature dependence of the thermal conductivity is stated as

$$k=T^2 \qquad (5.58)$$

The analytical solution for Eq. (5.56) is obtained as

$$T=\sqrt[3]{126x - 118} \qquad (5.59)$$

X	EXACT (T)	PRESENT (T)
1.1	2.741	2.741
2.0	5.117	5.117
3.0	6.383	6.383
4.0	7.281	7.281
4.9	7.934	7.934

Table 5.1: The exact solution and the numerical result

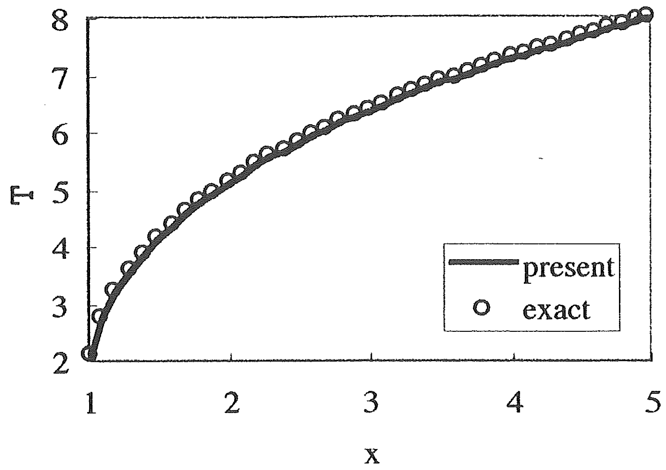

Figure 5.3: Distribution of Temperature

The analytical and numerical results are shown in Table 5.1 and Fig. 5.3. The numerical results are compared with the exact solutions. Table 5.1 and Fig. 5.3 show that they agreed with each other more than 4 figures.

In the second example, the governing equation and boundary conditions are the same as the first example.

The thermal conductivity is written as

$$k = T^2+3T+1 \tag{5.60}$$

X	EXACT (T)	PRESENT (T)
1.1	2.512	2.515
2.0	4.811	4.813
3.0	6.189	6.190
4.0	7.190	7.191
4.9	7.925	7.925

Table: 5.2 Comparison between analytical solutions and BEM results

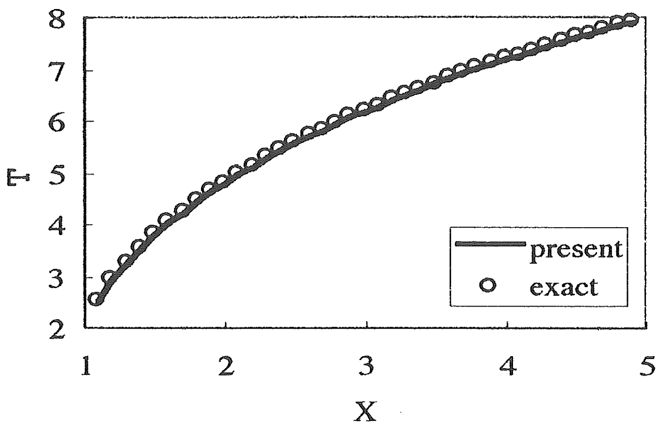

Figure 5.4: Distribution of Temperature

The exact solution is calculated as

$$2T^3 + 9T^2 + 6T - 396x + 332 = 0 \tag{5.61}$$

The comparison of exact and numerical results are shown as Table 5.2 and Fig. 5.4. The maximum error is 0.11%. It can be seen that the present results are accurate.

5.7 CONCLUSIONS

In this chapter, the new discrete integral approach has been proposed to handle the steady-state nonlinear heat conduction equation with temperature dependence of thermal conductivity by BEM. The main conclusions are summarized as follows:

1. The governing nonlinear heat conduction equation is transformed into a new form to which the discrete integral method is applicable. The new equation shall be solved instead of the original equation. For the volume integration, not the internal element but the internal discrete points and the boundary condition are put to use.
2. In the one-dimensional and the two-dimensional problems, the boundary integral equation are derived. A set of simultaneous equation are solved by the Newton-Raphson method.
3. This approach is not restricted by the temperature dependence of the heat conductivity.
4. In the one-dimensional problem the numerical results are obtained and compared with exact values. These results show that the numerical solutions have adequate accuracy.

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CHAPTER 6

CONCLUSIONS

In this thesis, a new approach is proposed for the functional approximation. The function is approximated by utilizing distribution of the delta function. Using this functional approximation the discrete integral method is developed for the general integral and the domain integral. This discrete integral method is employed in the bending problem of beam and the steady state heat conduction problem. The main work and conclusions are summarized as follows:

In chapter 1, a review of published literature and outline of this thesis are presented. Although the Galerkin tensor method is applied to elastic problems and gives accurate results, it can be used only for a limited range of the constant and linear forces. The dual reciprocity technique can solve a wide range of problems but usually requires a significant number of internal points or poles to represent the solution accurately, In order to avoid the requirement of the localized particular solution, the multiple reciprocity method is developed. This approach not only is employed in different types of body forces, but also is used for the other domain integral. However, this dual reciprocity method needs a sequence of higher order fundamental solutions. In the discrete integral method, the fundamental solutions of lower order is used for the treatment of the domain integral. This approach is simpler than MRM. Although the steady state nonlinear heat conduction equation with temperature dependence of thermal conductivity can be solved by Kirchhoff's transformation, this inverse transformation is complicated. When thermal conductivity is described only by linear, exponential and power functions in terms of temperature, a method by the use of the new variable can be employed in this problem. On the other hand, the discrete integral method can be applied to any kind of thermal conductivity and does not need the inverse Kirchhoff's transformation.

In chapter 2, the kind and characteristic of the BEM are described. The boundary element method has advantage over other numerical technique. The governing differential equation of Poisson problem in the two dimensions is transformed into integral equation using fundamental solution. The numerical implementation is discussed in detail. The quadratic shape functions and the quadratic surface element are used. The domain integral is evaluated by using a series of surface cells.

In chapter 3, the basic theory of the functional approximation utilizing the point source and the line source is described. In this theory the function is expressed by the boundary condition and the strength of the point source or the line source. Using the line source, the numerical results agrees with that by the exact values in the two dimensions. Using this functional approximation the discrete integral method is developed. The general integral or the domain integral can be expressed by the boundary integral and the sum of the values at the interior points. The discrete integral method using the line source is applied to the heat conduction equation with a heat source in the two dimensions by BEM. The numerical results by this approach agree with that by the classical BEM. This method uses the lower order fundamental solution so that it is simpler than MRM.

In chapter 4, The new algorithms without the dividing scheme are established for bending problems of the continuous beam including the inhomogeneous beam by the BEM. The conventional algorithms of the BEM for the bending problems of continuous beam are inefficient and have several points to be improved upon. Main points of this study are summarized as follows:

1. The formulation process is improved to derive the system of simultaneous equations. As a results, they are reformed to be much more compact.

2. The nondividing scheme is established contrary to the dividing scheme, which is needed in the conventional formulation on the occasions of simply supporting point, stepped beam, inhomogeneous beam (including discontinuously changeful rigidity) and concentrated

moment load.

3. A general scheme for treating the inhomogeneous beam including those with discontinuously changeful rigidity is established.

4. When an external force exists, the domain integral is performed by the discrete integral method. The domain integral is expressed by the boundary condition and the sum of some values at the internal points. The domain is not divided into cells.

From numerical examples, the rightness of the new algorithm is verified. These new algorithms greatly reduced the size of matrix as well as the computing time and, therefore, brought about high efficiency on the repetitive calculations. As a result, this will give a low cost for optimal design in daily work.

In chapter 5, In order to avoid the complex inverse transform using Kirchhoff's transformation, and the limitation of three cases of the thermal conductivity using a method by the use of the new variable, the discrete integral method is attempted to apply to the nonlinear heat conduction problem. The nonlinear equation is transformed into an equivalent form so that it is possible to apply the discrete integral method to this form. For the volume integral, the internal discrete points are put to use only. This new approach is suitable for any form of heat conductivity. This method is simpler and has the advantages of retaining the principal characteristics of BEM. The numerical results compared with exact results show that the numerical solutions have adequate accuracy in the one-dimensional problems. This problem will be carried out hereafter in the two and three dimensions.

In brief, the function has been approximated by utilizing distribution of the delta function. Using this functional approximation the discrete integral method has been proposed to treat the domain integral or the general integral and applied to the bending problem of beam and the heat conduction problem. Numerical calculation has been carried out and it is shown that results obtained by the proposed method have good convergency and adequate accuracy.

APPENDIX

(1) The feature of delta function

In this thesis, we usually use the delta function to deal with bending problem of beam and the other problems. In this appendix, we will explain the feature of delta function in detail. From Chapter 2, we obtain the equation of:

$$\frac{d^2 u^*(x,p)}{dx^2} = -\delta(x-p) \tag{2.5}$$

Where $\delta(x-p)$ is the Dirac delta function which is mathematically equivalent to the effect of a unit concentrated source applied at the point p. The fundamental solution is a particular solution of the adjoint form of the differential equation. Using the fundamental solution in the formulation, it is possible to treat a problem only on the boundary. The fundamental solution has an important relationship to Dirac's distribution. Dirac's distribution is a generalized function which has a sharp peek at x=p, and is zero except this point:

It is defined as

$$\delta(x-p) = \begin{cases} \infty & (x = p) \\ 0 & (x \neq p) \end{cases} \tag{A-1}$$

$$\int_V \delta(x-p)dV = \begin{cases} 1 & (p \in V) \\ 0 & (p \notin V) \end{cases} \tag{A-2}$$

$$\int_V f(x)\delta(x-p)dV = \begin{cases} f(p) & (p \in V) \\ 0 & (p \notin V) \end{cases} \tag{A-3}$$

Where $f(x)$ is a function which is continuous at $x = p$ and $f(p)$ is the value of $f(x)$ at the point $x = p$. Dirac's distribution can be explained in physical meanings. For example, in mechanics, it can be explained as a concentrated force acted on a point p of an infinite plate. This point is called source point. Against this, any point x in the field is called observation point. By using the feature of delta function expertly, we can readily derive the integral equation needed in the BEM.

(2) Indefinite integral including sign function

sgn is the sign function. When $x<p$, $\operatorname{sgn}(x-p)=-1$; When $x>p$, $\operatorname{sgn}(x-p)=1$.The following integal can be obtained with the sign function in simple style.

$$\int f(x)\operatorname{sgn}(x-p)dx = F(x)sgn(x-p)$$

$$-\int F(x)2\delta(x-p)dx$$

$$= F(x)sgn(x-p) - 2F(p)\int \delta(x-p)\,dx$$

$$=\{F(x) - F(p)\}\operatorname{sgn}(x-p) + c \qquad \text{(A-4)}$$

Where c is a constant. It is satisfied the following equation.

$$\int \delta(x-p)dx = \frac{1}{2}\operatorname{sgn}(x-p) + c \qquad \text{(A-5)}$$

In the bending problems of beam, the constant c is defined as zero.

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